

# FRACTIONAL SEMILINEAR NEUMANN PROBLEMS ARISING FROM A FRACTIONAL KELLER–SEGEL MODEL

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ABSTRACT. We consider the following fractional semilinear Neumann problem on a smooth bounded domain  $\Omega \subset \mathbb{R}^n$ ,  $n \geq 2$ ,

$$\begin{cases} (-\varepsilon\Delta)^{1/2}u + u = u^p, & \text{in } \Omega, \\ \partial_\nu u = 0, & \text{on } \partial\Omega, \\ u > 0, & \text{in } \Omega, \end{cases}$$

where  $\varepsilon > 0$  and  $1 < p < (n+1)/(n-1)$ . This is the fractional version of the semilinear Neumann problem studied by Lin–Ni–Takagi in the late 80’s. The problem arises by considering steady states of the Keller–Segel model with nonlocal chemical concentration diffusion. Using the semigroup language for the extension method and variational techniques, we prove existence of nonconstant smooth solutions for small  $\varepsilon$ , which are obtained by minimizing a suitable energy functional. In the case of large  $\varepsilon$  we obtain nonexistence of nonconstant solutions. It is also shown that as  $\varepsilon \rightarrow 0$  the solutions  $u_\varepsilon$  tend to zero in measure on  $\Omega$ , while they form spikes in  $\overline{\Omega}$ . The regularity estimates of the fractional Neumann Laplacian that we develop here are essential for the analysis. The latter results are of independent interest.

## 1. INTRODUCTION

In the famous paper [21] C.-S. Lin, W.-M. Ni and I. Takagi studied the existence of solutions to the one-parameter semilinear Neumann boundary value problem

$$(1.1) \quad \begin{cases} -\varepsilon\Delta u + u = g(u), & \text{in } \Omega, \\ \partial_\nu u = 0, & \text{on } \partial\Omega. \end{cases}$$

Here  $\Omega$  is a smooth bounded domain of  $\mathbb{R}^n$ ,  $n \geq 1$ ,  $\varepsilon$  is a positive parameter,  $\nu$  is the outer unit normal to  $\partial\Omega$  and  $g$  is a suitable nonnegative nonlinearity on  $\mathbb{R}$  vanishing for  $t \leq 0$ , growing superlinearly at infinity and such that, among other structural properties,  $g(t) = O(t^p)$  as  $t \rightarrow +\infty$ , where  $p > 1$  is below the *critical* Sobolev exponent  $(n+2)/(n-2)$  when  $n \geq 3$ . In particular, the authors study the existence of positive weak solutions to (1.1) by applying the Mountain Pass Lemma of A. Ambrosetti and P. Rabinowitz [3] to the energy functional

$$\mathcal{J}_\varepsilon(u) = \frac{1}{2} \int_\Omega (\varepsilon |\nabla u|^2 + u^2) dx - \int_\Omega G(u) dx,$$

where  $G$  is an antiderivative of  $g$ . It is proved in [21, Theorem 2] that if  $\varepsilon$  is small enough, there exists a positive smooth solution  $u_\varepsilon$  (a *critical least energy solution*) for which

$$\mathcal{J}_\varepsilon(u_\varepsilon) \leq C\varepsilon^{n/2},$$

for a positive constant  $C$  which does not depend on  $\varepsilon$ . This property allows to prove that the family of solutions  $\{u_\varepsilon\}_{\varepsilon>0}$  is uniformly bounded for small  $\varepsilon$  and to obtain the convergence of such solutions to 0 in measure when  $\varepsilon \rightarrow 0$ , see [21, Corollary 2.1]. Actually, for the nonlinearity  $g(t) = t^p$  in [21] it is shown that the boundedness of  $\{u_\varepsilon\}_{\varepsilon>0}$  holds for all  $\varepsilon > 0$  and that  $u \equiv 1$  is the only positive solution to problem (1.1) for large  $\varepsilon$ . Besides, in [21, Proposition 4.1] the authors exhibit a striking property concerning the shape of the graphs of the solutions  $\{u_\varepsilon\}_{\varepsilon>0}$  for small  $\varepsilon$ , which actually look

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like spikes in  $\overline{\Omega}$ . Such a property was the starting point of the research continued in the papers [24, 25] concerning mainly the localization of these spikes on the boundary of  $\Omega$ .

The aim of the present paper is to extend all the previous cited results to the problem

$$(1.2) \quad \begin{cases} (-\varepsilon\Delta)^{1/2}u + u = g(u), & \text{in } \Omega, \\ \partial_\nu u = 0, & \text{on } \partial\Omega, \end{cases}$$

where  $\Omega$  is as before and  $\varepsilon > 0$ . The nonlinearity  $g$  is defined as

$$(1.3) \quad g(t) = \begin{cases} t^p, & \text{if } t \geq 0, \\ 0, & \text{if } t \leq 0. \end{cases}$$

for

$$(1.4) \quad 1 < p < \frac{n+1}{n-1}.$$

Notice that  $(n+1)/(n-1)$  is the critical Sobolev *trace* exponent. The operator  $(-\varepsilon\Delta)^{1/2}$  is understood as the square root of the Laplacian in the bounded domain  $\Omega$  encoding the homogeneous Neumann boundary condition, that is, the *fractional Neumann Laplacian* which is defined as follows. Let  $\{\varphi_k\}_{k \in \mathbb{N}_0}$  and  $\{\lambda_k\}_{k \in \mathbb{N}_0}$  be the eigenfunctions and eigenvalues of the Laplacian  $-\Delta$  on  $\Omega$  with homogeneous Neumann boundary condition. The  $(\varepsilon-)$ Neumann Laplacian is the operator that acts on an  $L^2$  function  $u(x) = \sum_{k=0}^{\infty} u_k \varphi_k(x)$  as

$$-\varepsilon\Delta_N u(x) = \sum_{k=0}^{\infty} (\varepsilon\lambda_k) u_k \varphi_k(x),$$

in a suitable sense. Then the fractional  $(\varepsilon-)$ Neumann Laplacian is given by

$$(-\varepsilon\Delta_N)^{1/2} u(x) = \sum_{k=0}^{\infty} (\varepsilon\lambda_k)^{1/2} u_k \varphi_k(x).$$

For more details see Sections 2 and 3.

Using the language of semigroups as introduced in [31], see also [32], one can check that  $(-\varepsilon\Delta_N)^{1/2}$  is indeed a nonlocal operator. In fact,

$$(-\varepsilon\Delta_N)^{1/2} u(x) = \frac{1}{\sqrt{\pi}} \int_0^\infty (e^{t\varepsilon\Delta_N} u(x) - u(x)) \frac{dt}{t^{3/2}}, \quad x \in \Omega,$$

where  $e^{t\varepsilon\Delta_N} u(x)$  is the heat diffusion semigroup generated by the Neumann Laplacian acting on  $u$ . Then, by following the ideas of [31, 32], one can conclude that for a smooth function  $u$  we have the pointwise integro-differential formula

$$(-\varepsilon\Delta_N)^{1/2} u(x) = c_{n,\Omega} \text{P.V.} \int_\Omega (u(x) - u(z)) K(x, z) dz, \quad x \in \Omega,$$

where  $c_{n,\Omega}$  is a positive constant and the kernel  $K(x, z)$ , given in terms of the heat kernel for the Neumann Laplacian, satisfies the estimate  $K(x, z) \sim \varepsilon^{1/2} |x - z|^{-(n+1)}$ , for  $x, z \in \Omega$ . Moreover, the fundamental solution of  $(-\varepsilon\Delta_N)^{1/2}$  behaves like  $\varepsilon^{1/2} |x - z|^{-(n-1)}$ . We will not go further into these details here.

Looking at problem (1.2) and taking into account the nonlocality of the fractional Neumann Laplacian, a considerable issue that arises is to provide a suitable definition of weak solution. The answer relies in understanding  $(-\varepsilon\Delta_N)^{1/2} u$  as the normal derivative on  $\Omega$  of a proper harmonic extension of  $u$  to the cylinder  $\mathcal{C} := \Omega \times (0, \infty)$ . Let us first explain this idea in the classical case. It is known that if  $v(x, y) : \mathbb{R}^n \times (0, \infty) \rightarrow \mathbb{R}$  is the harmonic extension of a smooth function  $u : \mathbb{R}^n \rightarrow \mathbb{R}$  to the upper half space, namely, if  $v$  is the solution to

$$\begin{cases} \Delta_x v + v_{yy} = 0, & \text{in } \mathbb{R}^n \times (0, \infty), \\ v(x, 0) = u(x), & \text{on } \mathbb{R}^n, \end{cases}$$

then

$$-v_y(x, 0) = (-\Delta)^{1/2}u(x), \quad x \in \mathbb{R}^n,$$

where  $(-\Delta)^{1/2}$  is the square root of the Laplacian on  $\mathbb{R}^n$ . Such an identity can be checked by using the Fourier transform on the variable  $x$ . We can understand this last property by using the language of the semigroups as explained in [31, 32]. Indeed, the solution  $v$  above is the *Poisson semigroup* generated by the Laplacian on  $\mathbb{R}^n$ , which can be written in terms of the classical heat semigroup  $e^{t\Delta}$  via the so called *Bochner subordination formula*:

$$v(x, y) \equiv e^{-y(-\Delta)^{1/2}}u(x) = \frac{y}{2\sqrt{\pi}} \int_0^\infty e^{-y^2/(4t)} e^{t\Delta}u(x) \frac{dt}{t^{3/2}}.$$

Then,

$$-v_y(x, y) = (-\Delta)^{1/2}e^{-y(-\Delta)^{1/2}}u(x), \quad y > 0, \quad \text{and} \quad -v_y(x, y)|_{y=0} = (-\Delta)^{1/2}u(x).$$

The advantage now is that nothing prevents us to replace  $-\Delta$  in the formulas for the semigroup above by a positive operator  $L$  (defined together with its functional domain in order to include boundary conditions), so the same characterization for  $L^{1/2}$  as a Dirichlet-to-Neumann operator holds, see [31, 32], or [13] for more general operators. Moreover, this semigroup language avoids the use of the Fourier transform, which is not available for a general differential operator  $L$ . In our particular case, the result for the  $\varepsilon$ -fractional Neumann Laplacian reads as follows. If  $v(x, y)$  is the solution to

$$\begin{cases} \varepsilon \Delta_x v + v_{yy} = 0, & \text{in } \mathcal{C} := \Omega \times (0, \infty), \\ \partial_\nu v = 0, & \text{on } \partial_L \mathcal{C} := \partial\Omega \times [0, \infty), \\ v(x, 0) = u(x), & \text{on } \Omega, \end{cases}$$

then

$$-v_y(x, 0) = (-\varepsilon \Delta_N)^{1/2}u(x), \quad x \in \Omega.$$

See Section 2 for more details.

Going back to our problem (1.2), we can define a weak solution  $u$  as the trace over  $\Omega$  of a weak solution  $v$  to the problem with semilinear Neumann boundary condition

$$(1.5) \quad \begin{cases} \varepsilon \Delta_x v + v_{yy} = 0, & \text{in } \mathcal{C}, \\ \partial_\nu v = 0, & \text{on } \partial_L \mathcal{C}, \\ -v_y(x, 0) + v(x, 0) = g(v(x, 0)), & \text{on } \Omega. \end{cases}$$

that is

$$u(x) = v(x, 0), \quad x \in \Omega.$$

The energy functional related to (1.5) is

$$(1.6) \quad \mathcal{I}_\varepsilon(v) = \frac{1}{2} \iint_{\mathcal{C}} (\varepsilon |\nabla_x v|^2 + v_y^2) dx dy + \frac{1}{2} \int_\Omega v(x, 0)^2 dx - \int_\Omega G(v(x, 0)) dx,$$

where  $G$  is an antiderivative of the nonlinearity  $g$  given in (1.3). Here we can apply the techniques of the Calculus of Variations to prove existence of critical points of  $\mathcal{I}_\varepsilon$ , yielding nonconstant regular solutions  $u$  to (1.2).

The present paper arises from a concrete application that can also be seen as a nonlocal version of the model where the investigation in [21] took its origin. Problem (1.2) appears when considering the steady states of the Keller–Segel system when the diffusion of the concentration of the chemical is nonlocal. Indeed, while [21] is involved in the study of stationary solutions to the classical Keller–Segel chemotaxis model system posed in  $\Omega$ , we look the solutions to (1.2) as steady states to the following local-nonlocal system

$$(1.7) \quad \begin{cases} \rho_t - D_1 \Delta \rho + \chi \nabla \cdot (\rho \nabla \log c) = 0, & \text{in } \Omega \times (0, \infty), \\ c_t + D_2 (-\Delta)^{1/2} c + ac - b\rho = 0, & \text{in } \Omega \times (0, \infty), \\ \partial_\nu \rho = \partial_\nu c = 0, & \text{on } \partial\Omega \times (0, \infty), \end{cases}$$

for some positive constants  $D_1, D_2, \chi, a, b$ , with the initial conditions

$$(1.8) \quad \begin{cases} \rho(x, 0) = \rho_0(x), & \text{on } \Omega, \\ c(x, 0) = c_0(x), & \text{on } \Omega. \end{cases}$$

Here  $\rho(x, t)$  describes the density of a bacteria population (such as amoebae) and  $c(x, t)$  is the density concentration of a certain chemical. In the model (1.7)–(1.8) the diffusion is assumed to be local for the bacteria, while it is nonlocal for the chemical. Due to the conservation of mass property for the first equation in (1.7), a steady state for the system is a couple of functions  $u, v$  satisfying

$$(1.9) \quad \begin{cases} D_1 \Delta u - \chi \nabla \cdot (u \nabla \log v) = 0, & \text{in } \Omega, \\ D_2 (-\Delta)^{1/2} v + av - bu = 0, & \text{in } \Omega, \\ \partial_\nu u = \partial_\nu v = 0, & \text{on } \partial\Omega, \end{cases}$$

with the condition

$$(1.10) \quad u_\Omega := \frac{1}{|\Omega|} \int_\Omega u(x) dx$$

equal to an assigned positive constant  $\bar{u}$ . Of course  $u = \bar{u}, v = a^{-1}b\bar{u}$  is a solution to (1.9). Thus it makes sense to look for positive nonconstant solutions. Arguing similarly to [21], it is possible to write  $u = \lambda v^{\chi/D_1}$  for some positive constant  $\lambda$ . Then the system (1.9) is equivalent to find a solution to (1.4), where  $\varepsilon = D_2/a$ ,  $g(t) = t^p$  for  $t \geq 0$  and  $p = \chi/D_1$  and  $v_\Omega = \bar{v}$ .

Since the appearance of the papers by L. Caffarelli, L. Silvestre and collaborators [6, 7, 8, 9, 27, 28, 29] nonlocal PDEs with fractional Laplacians became a topic which is nowadays deserving a lot of attention, also because of the various applications in several fields. As far as the Neumann Laplacian is concerned, some problems were studied in [2, 19, 22]. A fractional Keller–Segel model was considered in [11], though the author there proposes to model the concentration of the chemical with the usual local diffusion while the bacteria satisfy a nonlocal diffusion in dimension one. This is in contrast with our model.

Notice that we could also model the diffusion of the chemical with a fractional Neumann Laplacian with power different than  $1/2$ . The extension problem for such an operator is available (see the generalization of the Caffarelli–Silvestre result of [7] given in [31, 32]) so in principle part of the analysis could be carried out. This generalization will appear elsewhere. The nonlinearity (1.3) which we use could be replaced by a more general nonlinearity  $g(x, t)$  under suitable structural conditions as in [21] without affecting the statements of the results or the techniques needed. We stick to (1.3) to keep a clean presentation of the ideas.

The paper is organized as follows. In Section 2 we give the functional framework which is necessary for the analysis of (1.2). In particular, by making use of the extension problem (Theorem 2.1) and the efficient semigroup language approach, we identify the domain of the operator  $(-\varepsilon \Delta_N)^{1/2}$  (Theorem 2.4) and exhibit a trace inequality (Lemma 2.5), valid for functions belonging to a suitable functional space on the cylinder  $\mathcal{C}$ . All this will serve to give sense to the definition of weak solution for the semilinear problem (1.5). Section 3 is entirely devoted to discuss existence (Lemma 3.3), Harnack estimates (Theorem 3.4) and regularity (Theorem 3.5) of weak solutions to the linear problem

$$(1.11) \quad \begin{cases} (-\varepsilon \Delta)^{1/2} u + u = f, & \text{in } \Omega, \\ \partial_\nu u = 0, & \text{on } \partial\Omega. \end{cases}$$

We point out that for the equation  $(-\Delta_N)^{1/2} u = f$  parallel results can be found in [22], while for the case of the fractional Dirichlet Laplacian  $(-\Delta_D)^{1/2} u = f$  one can see [5]. Section 4 introduces the proper concept of weak solution for the semilinear problem (1.2) and shows the existence of a positive nonconstant smooth solution  $u_\varepsilon$  for small  $\varepsilon$ . Its extension  $v_\varepsilon$  is obtained as a critical point of the energy functional (1.6) associated to problem (1.5) (Theorem 4.3, Corollary 4.4). In Section 5 we employ a Moser iteration argument to obtain, for small  $\varepsilon > 0$ , an  $L^\infty$  bound of the sequence  $\{u_\varepsilon\}_{\varepsilon>0}$  (Theorem 5.1). From here an  $L^q$  estimate of each  $u_\varepsilon$  is derived (Lemma 5.4). This information allows to describe the geometry of the functions  $\{u_\varepsilon\}_{\varepsilon>0}$ , that are shown to be spike solutions (Theorem

5.2). In Theorem 5.5 we use the previous results and a blow-up argument to get the boundedness of any solution  $\{u_\varepsilon\}_{\varepsilon>0}$  independent of  $\varepsilon$ . Finally, Theorem 5.6 concludes the paper by showing the nonexistence of nonconstant positive solutions to (1.2) for large  $\varepsilon$ .

Throughout the paper  $C, C_0, c$  denote positive constants. Subscripts in the constants point out the dependence on a group of parameters. By  $\Omega$  we denote a smooth bounded domain of  $\mathbb{R}^n$ ,  $n \geq 2$ . For positive numbers  $A, B$  the symbol  $A \sim B$  means that for positive constants  $c, C$  we have  $cA \leq B \leq CA$ , and we call  $c, C$  the equivalence constants.

## 2. EXTENSION PROBLEM AND DOMAIN FOR $(-\varepsilon\Delta_N)^{1/2}$

We denote by  $\langle \cdot, \cdot \rangle_{L^2(\Omega)}$ ,  $\langle \cdot, \cdot \rangle_{H^1(\Omega)}$  the scalar products in  $L^2(\Omega)$  and in the usual Sobolev space  $H^1(\Omega)$ , respectively, and  $\langle \cdot, \cdot \rangle$  will mean the pairing between a Hilbert space and its dual. Consider the homogeneous Neumann eigenvalue problem for the Laplacian in  $\Omega$ :

$$(2.1) \quad \begin{cases} -\Delta\varphi = \lambda\varphi, & \text{in } \Omega, \\ \partial_\nu\varphi = 0, & \text{on } \partial\Omega, \end{cases}$$

for  $\lambda \geq 0$ . It is well known (see for example [12, 15]) that there exists a sequence of nonnegative eigenvalues  $\{\lambda_k\}_{k \in \mathbb{N}_0}$  that corresponds to eigenfunctions  $\{\varphi_k\}_{k \in \mathbb{N}_0}$  in  $H^1(\Omega)$  which are weak solutions to (2.1). We have that  $\lambda_0 = 0$ ,  $\varphi_0 = 1/\sqrt{|\Omega|}$ ,  $\int_\Omega \varphi_k dx = 0$ , for all  $k \geq 1$  and each  $\varphi_k$  belongs to  $C^\infty(\overline{\Omega})$ . The eigenfunctions  $\{\varphi_k\}_{k \in \mathbb{N}_0}$  form an orthonormal basis of  $L^2(\Omega)$ . By using the  $L^2$  normalization and the weak formulation of the equation we see that  $\|\varphi_k\|_{H^1(\Omega)}^2 = 1 + \lambda_k$ . It is easy to check that  $\{\varphi_k\}_{k \in \mathbb{N}_0}$  is also an orthogonal basis of  $H^1(\Omega)$ . Hence, since  $\langle u, \varphi_k \rangle_{H^1(\Omega)} = (1 + \lambda_k) \langle u, \varphi_k \rangle_{L^2(\Omega)}$ , we find

$$(2.2) \quad H^1(\Omega) = \left\{ u \in L^2(\Omega) : \|u\|_{H^1(\Omega)}^2 = \sum_{k=0}^{\infty} (1 + \lambda_k) |\langle u, \varphi_k \rangle_{L^2(\Omega)}|^2 < \infty \right\}.$$

For a function  $u \in H^1(\Omega)$ , we define the (negative) Neumann Laplacian of  $u$  as an element  $-\Delta_N u$  in the dual space  $H^1(\Omega)'$  verifying

$$(2.3) \quad \langle -\Delta_N u, v \rangle = \int_\Omega \nabla u \cdot \nabla v dx, \quad \text{for every } v \in H^1(\Omega).$$

In terms of the orthogonal basis  $\{\varphi_k\}_{k \in \mathbb{N}_0}$  we can write

$$-\Delta_N u = \sum_{k=1}^{\infty} \lambda_k \langle u, \varphi_k \rangle_{L^2(\Omega)} \varphi_k, \quad \text{in } H^1(\Omega)',$$

namely, for each  $v \in H^1(\Omega)$  we have

$$(2.4) \quad \langle -\Delta_N u, v \rangle = \sum_{k=1}^{\infty} \lambda_k \langle u, \varphi_k \rangle_{L^2(\Omega)} \langle v, \varphi_k \rangle_{L^2(\Omega)}.$$

Indeed, by the weak formulation of the eigenvalue problem,  $\langle -\Delta_N u, \varphi_k \rangle = \lambda_k \langle u, \varphi_k \rangle_{L^2(\Omega)}$ , for all  $k \geq 0$ , and the identity (2.4) follows by linearity and density. Notice that  $-\Delta_N$  has a nontrivial kernel in  $H^1(\Omega)$ , namely, the set of constant functions.

In order to define the fractional  $\varepsilon$ -Neumann Laplacian  $(-\varepsilon\Delta_N)^{1/2}$  for each  $\varepsilon > 0$ , we consider the Hilbert space

$$\mathcal{H}_\varepsilon(\Omega) \equiv \text{Dom}((-\varepsilon\Delta_N)^{1/2}) := \left\{ u \in L^2(\Omega) : \sum_{k=1}^{\infty} (\varepsilon\lambda_k)^{1/2} |\langle u, \varphi_k \rangle_{L^2(\Omega)}|^2 < \infty \right\},$$

under the scalar product

$$\langle u, v \rangle_{\mathcal{H}_\varepsilon(\Omega)} := \langle u, v \rangle_{L^2(\Omega)} + \varepsilon^{1/2} \sum_{k=1}^{\infty} \lambda_k^{1/2} \langle u, \varphi_k \rangle_{L^2(\Omega)} \langle v, \varphi_k \rangle_{L^2(\Omega)},$$

so that the norm in  $\mathcal{H}_\varepsilon(\Omega)$  is given by

$$\|u\|_{\mathcal{H}_\varepsilon(\Omega)}^2 = \|u\|_{L^2(\Omega)}^2 + \varepsilon^{1/2} \sum_{k=1}^{\infty} \lambda_k^{1/2} |\langle u, \varphi_k \rangle_{L^2(\Omega)}|^2.$$

Since  $\lambda_k \nearrow \infty$ , it is obvious that  $C^\infty(\overline{\Omega}) \subset H^1(\Omega) \subset \mathcal{H}_\varepsilon(\Omega)$ . For  $u \in \mathcal{H}_\varepsilon(\Omega)$ , we define  $(-\varepsilon\Delta_N)^{1/2}u$  as an element in the dual space  $\mathcal{H}_\varepsilon(\Omega)'$  given by

$$(-\varepsilon\Delta_N)^{1/2}u = \sum_{k=1}^{\infty} (\varepsilon\lambda_k)^{1/2} \langle u, \varphi_k \rangle_{L^2(\Omega)} \varphi_k, \quad \text{in } \mathcal{H}_\varepsilon(\Omega)',$$

that is, for any function  $v \in \mathcal{H}_\varepsilon(\Omega)$ ,

$$\langle (-\varepsilon\Delta_N)^{1/2}u, v \rangle = \sum_{k=1}^{\infty} (\varepsilon\lambda_k)^{1/2} \langle u, \varphi_k \rangle_{L^2(\Omega)} \langle v, \varphi_k \rangle_{L^2(\Omega)}.$$

As before, the set of constant functions is the nontrivial kernel of  $(-\varepsilon\Delta_N)^{1/2}$  in  $\mathcal{H}_\varepsilon(\Omega)$ . The last identity can be rewritten, in a parallel way to (2.3), as

$$\langle (-\varepsilon\Delta_N)^{1/2}u, v \rangle = \int_{\Omega} (-\varepsilon\Delta_N)^{1/4} u (-\varepsilon\Delta_N)^{1/4} v \, dx, \quad \text{for every } v \in \mathcal{H}_\varepsilon(\Omega),$$

where  $(-\varepsilon\Delta_N)^{1/4}$  is defined by taking the power  $1/4$  of the eigenvalues  $\lambda_k$ .

Before presenting the extension problem for  $(-\varepsilon\Delta_N)^{1/2}$  we collect some well known facts about the heat equation on  $\Omega$  with Neumann boundary condition. Let  $u \in L^p(\Omega)$ ,  $1 \leq p \leq \infty$ . Then the unique classical solution to the Neumann heat equation

$$\begin{cases} w_t = \Delta w, & \text{in } \Omega \times (0, \infty), \\ \partial_\nu w = 0, & \text{on } \partial\Omega \times [0, \infty), \\ w(x, 0) = u(x), & \text{on } \Omega, \end{cases}$$

is given by

$$w(x, t) \equiv e^{t\Delta_N} u(x) = \int_{\Omega} \mathcal{W}_t(x, z) u(z) \, dz,$$

where

$$\mathcal{W}_t(x, z) = \sum_{k=0}^{\infty} e^{-t\lambda_k} \varphi_k(x) \varphi_k(z),$$

is the (distributional) Neumann heat kernel. We have

$$(2.5) \quad \int_{\Omega} \mathcal{W}_t(x, z) \, dz = 1, \quad \text{for all } x \in \Omega, \, t > 0.$$

Moreover, for each  $T > 0$  there exist positive constants  $c_1, c_2, c_3, c_4, c, C$ , depending only on  $\Omega, n$  and  $T$ , such that

$$(2.6) \quad c_1 \frac{e^{-\frac{|x-z|^2}{c_2 t}}}{t^{n/2}} \leq \mathcal{W}_t(x, z) \leq c_3 \frac{e^{-\frac{|x-z|^2}{c_4 t}}}{t^{n/2}}, \quad \text{for all } x, z \in \Omega, \, 0 < t < T,$$

and

$$(2.7) \quad |\nabla_x \mathcal{W}_t(x, z)| \leq \frac{C}{t^{(n+1)/2}} e^{-c \frac{|x-z|^2}{t}}, \quad \text{for all } x, z \in \Omega, \, t > 0.$$

Moreover, there is a constant  $M$  such that  $|\mathcal{W}_t(x, z)| \leq M$  for all  $x, z \in \Omega$  and  $t > T$ . For these properties see [16, 26, 33, 34, 35].

We particularize to  $(-\varepsilon\Delta_N)^{1/2}$  the general extension problem proved in [31, 32].

**Theorem 2.1** (Extension problem for  $(-\varepsilon\Delta_N)^{1/2}$ ). *Let  $u \in \mathcal{H}_\varepsilon(\Omega)$  such that  $\int_\Omega u \, dx = 0$ . Define*

$$(2.8) \quad v(x, y) = e^{-y(-\varepsilon\Delta_N)^{1/2}} u(x) := \sum_{k=1}^{\infty} e^{-y(\varepsilon\lambda_k)^{1/2}} \langle u, \varphi_k \rangle_{L^2(\Omega)} \varphi_k(x).$$

*Then  $v \in H^1(\mathcal{C})$  is the unique weak solution to the extension problem*

$$(2.9) \quad \begin{cases} \varepsilon\Delta_x v + v_{yy} = 0, & \text{in } \mathcal{C}, \\ \partial_\nu v = 0, & \text{on } \partial_L \mathcal{C}, \\ v(x, 0) = u(x), & \text{on } \Omega, \end{cases}$$

*where  $\nu$  is the outward normal to the lateral boundary  $\partial_L \mathcal{C}$  of  $\mathcal{C}$ , such that  $\int_\Omega v(x, y) \, dx = 0$ , for all  $y \geq 0$ . More precisely,*

$$\iint_{\mathcal{C}} (\varepsilon \nabla_x v \cdot \nabla_x \psi + v_y \psi_y) \, dx \, dy = 0,$$

*for all test functions  $\psi \in H^1(\mathcal{C})$  with zero trace over  $\Omega$ ,  $\text{tr}_\Omega \psi = 0$ , and  $\lim_{y \rightarrow 0^+} v(x, y) = u(x)$  in  $L^2(\Omega)$ . Furthermore, the function  $v$  is the unique minimizer of the energy functional*

$$(2.10) \quad \mathcal{F}(v) = \frac{1}{2} \iint_{\mathcal{C}} (\varepsilon |\nabla_x v|^2 + |v_y|^2) \, dx \, dy,$$

*over the set  $\mathcal{U} = \{v \in H^1(\mathcal{C}) : \text{tr}_\Omega v = u\}$ . We can also write*

$$(2.11) \quad v(x, y) = \frac{\varepsilon^{1/2} y}{2\sqrt{\pi}} \int_0^\infty e^{-\varepsilon y^2/(4t)} e^{t\Delta_N} u(x) \frac{dt}{t^{3/2}} = \int_\Omega \mathcal{P}_{\varepsilon^{1/2}y}(x, z) u(z) \, dz,$$

*where, for any  $y > 0$ ,*

$$(2.12) \quad \mathcal{P}_y(x, z) = \sum_{k=0}^{\infty} e^{-y\lambda_k^{1/2}} \varphi_k(x) \varphi_k(z) = \frac{y}{2\sqrt{\pi}} \int_0^\infty e^{-y^2/(4t)} \mathcal{W}_t(x, z) \frac{dt}{t^{3/2}},$$

*is the Neumann–Poisson kernel. An equivalent formula for  $v$  is*

$$v(\cdot, y) = \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-\varepsilon y^2/(4t)} e^{t\Delta_N} ((-\Delta_N)^{1/2} u) \frac{dt}{t^{1/2}}, \quad \text{in } \mathcal{H}_\varepsilon(\Omega)'.$$

*Moreover,*

$$(2.13) \quad -\lim_{y \rightarrow 0^+} v_y = (-\varepsilon\Delta_N)^{1/2} u, \quad \text{in } \mathcal{H}_\varepsilon(\Omega)'.$$

*Proof.* From [31, 32] we know that  $v \in C^\infty((0, \infty); H^1(\Omega)) \cap C([0, \infty); L^2(\Omega))$ . Observe that

$$\iint_{\mathcal{C}} v^2 \, dx \, dy = \sum_{k=1}^{\infty} \int_0^\infty e^{-2y(\varepsilon\lambda_k)^{1/2}} |\langle u, \varphi_k \rangle_{L^2(\Omega)}|^2 \, dy \leq C \|u\|_{L^2(\Omega)}^2,$$

and

$$(2.14) \quad \begin{aligned} \iint_{\mathcal{C}} (\varepsilon |\nabla_x v|^2 + v_y^2) \, dx \, dy &= 2 \sum_{k=1}^{\infty} \varepsilon \lambda_k |\langle u, \varphi_k \rangle_{L^2(\Omega)}|^2 \int_0^\infty e^{-2y(\varepsilon\lambda_k)^{1/2}} \, dy \\ &= \sum_{k=1}^{\infty} (\varepsilon \lambda_k)^{1/2} |\langle u, \varphi_k \rangle_{L^2(\Omega)}|^2 = \|(-\varepsilon\Delta_N)^{1/4} u\|_{L^2(\Omega)}^2. \end{aligned}$$

Therefore  $v \in H^1(\mathcal{C})$ . Let  $\psi \in H^1(\mathcal{C})$  such that  $\text{tr}_\Omega \psi = 0$ . For almost every  $y > 0$  we can write

$$\psi(x, y) = \sum_{k=0}^{\infty} \langle \psi(\cdot, y), \varphi \rangle_{L^2(\Omega)} \varphi_k(x) =: \sum_{k=0}^{\infty} \psi_k(y) \varphi_k(x).$$

Then, by using (2.4) and the definition of  $v$ , for almost every  $y > 0$  we have

$$\int_{\Omega} v_{yy} \psi \, dx = \sum_{k=1}^{\infty} \varepsilon \lambda_k e^{-y(\varepsilon \lambda_k)^{1/2}} \langle u, \varphi_k \rangle_{L^2(\Omega)} \psi_k(y) = \varepsilon \int_{\Omega} \nabla_x v \cdot \nabla_x \psi \, dx.$$

Integrating this identity in  $y$  and applying integration by parts,

$$\iint_{\mathcal{C}} (\varepsilon \nabla_x v \cdot \nabla_x \psi + v_y \psi_y) \, dx \, dy = - \lim_{y \rightarrow 0^+} \int_{\Omega} v_y(x, y) \psi(x, y) \, dx = 0.$$

Hence  $v$  in the statement is a weak solution to (2.9).

Uniqueness can be proved in two ways. One is by writing  $v(x, y) = \sum_{k=0}^{\infty} v_k(y) \varphi_k(x)$  and showing that each  $v_k(y)$  is the unique solution to an ordinary differential equation, see [31, 32]. Another way of proving uniqueness is by using the weak formulation for the difference  $V$  of two solutions to (3.3). Indeed, using  $V$  itself as a test function in the weak formulation, we get that  $\nabla_{x,y} V = 0$ . But then, since  $V \in H^1(\mathcal{C})$ , we must have  $V \equiv 0$  on  $\mathcal{C}$ .

Concerning the minimization property of  $v$ , let us take any other function  $w \in \mathcal{U}$  and use  $v - w$  as a test function in the weak formulation of (2.9). Then we get

$$\mathcal{F}(v) = \frac{1}{2} \iint_{\mathcal{C}} (\varepsilon \nabla_x v \cdot \nabla_x w + v_y w_y) \, dx \, dy.$$

By using the Cauchy–Schwarz and Young inequalities we have  $\mathcal{F}(v) \leq \mathcal{F}(w)$ , and the uniqueness follows from the strict convexity of  $\mathcal{F}$ . The rest of the formulas in the statement, as well as (2.13), follow from [32, Theorem 1.1].  $\square$

**Definition 2.2.** For any  $u \in \mathcal{H}_{\varepsilon}(\Omega)$  such that  $\int_{\Omega} u \, dx = 0$ , we will call the solution  $v$  to problem (2.9) the  $\varepsilon$ -Neumann harmonic extension of  $u$  and we write  $v = E^{\varepsilon}(u)$ .

**Remark 2.3** (Extensions of functions with nonzero average). Observe that if  $\int_{\Omega} u \, dx \neq 0$  then the function

$$(2.15) \quad v(x, y) = \sum_{k=0}^{\infty} e^{-y(\varepsilon \lambda_k)^{1/2}} \langle u, \varphi_k \rangle_{L^2(\Omega)} \varphi_k(x).$$

is not in  $L^2(\mathcal{C})$  in general but only its gradient  $\nabla v \in L^2(\mathcal{C})$ . In order to give a suitable definition of the  $\varepsilon$ -Neumann harmonic extension of  $u$ , we first solve the extension problem (2.9) with initial data  $\tilde{u} = u - u_{\Omega}$ , where  $u_{\Omega}$  denotes the integral average of  $u$  over  $\Omega$ , see (1.10), in order to find a function  $\tilde{v} = E^{\varepsilon}(\tilde{u})$ . Then we define

$$v \equiv E^{\varepsilon}(u) := E^{\varepsilon}(\tilde{u}) + u_{\Omega},$$

which clearly coincides with (2.15).

Using the fact that the fractional  $\varepsilon$ -Neumann Laplacian does not see constants, we have

$$(-\varepsilon \Delta_N)^{1/2} u = (-\varepsilon \Delta_N)^{1/2} \tilde{u} = - \lim_{y \rightarrow 0^+} \tilde{v}_y = - \lim_{y \rightarrow 0^+} v_y, \quad \text{in } \mathcal{H}_{\varepsilon}(\Omega)'.$$

Notice that the extension result gives in particular that

$$(2.16) \quad u \in \mathcal{H}_{\varepsilon}(\Omega), \quad \int_{\Omega} u \, dx = 0, \quad \text{implies} \quad u = \text{tr}_{\Omega} v, \quad \text{for some } v \in H^1(\mathcal{C}), \quad \int_{\Omega} v(x, y) \, dx = 0, \quad y \geq 0.$$

Now two questions come into our attention:

- (i) Is the opposite implication in (2.16) true?
- (ii) Is there any integral characterization of the space  $\mathcal{H}_{\varepsilon}(\Omega)$ ?

The answer to question (i) is affirmative, and it will follow by an application of the trace inequality of Lemma 2.5. The answer to the second question is the identity  $\mathcal{H}_{\varepsilon}(\Omega) = H^{1/2}(\Omega)$ , see Theorem 2.4.



**2.1. Characterization of  $\mathcal{H}_\varepsilon(\Omega) = \text{Dom}((-\varepsilon\Delta_N)^{1/2})$ .** Consider next the following fractional Sobolev space on the basis  $\Omega \times \{0\}$  of  $\mathcal{C}$ :

$$H^{1/2}(\Omega) = \left\{ u \in L^2(\Omega) : \|u\|_{H^{1/2}(\Omega)}^2 := \|u\|_{L^2(\Omega)}^2 + [u]_{H^{1/2}(\Omega)}^2 < \infty \right\},$$

where

$$[u]_{H^{1/2}(\Omega)}^2 := \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{n+1}} dx dy.$$

We denote by  $H^{-1/2}(\Omega)$  the dual space of the Hilbert space  $H^{1/2}(\Omega)$ .

Using again the semigroup language we can prove the following result.

**Theorem 2.4** (Domain of  $(-\varepsilon\Delta_N)^{1/2}$ ). *For each  $\varepsilon > 0$  we have*

$$\mathcal{H}_\varepsilon(\Omega) = H^{1/2}(\Omega),$$

as Hilbert spaces. In particular,

$$\|(-\varepsilon\Delta_N)^{1/4}u\|_{L^2(\Omega)}^2 = \sum_{k=1}^{\infty} (\varepsilon\lambda_k)^{1/2} |\langle u, \varphi_k \rangle_{L^2(\Omega)}|^2 \sim \varepsilon^{1/2} [u]_{H^{1/2}(\Omega)}^2,$$

with equivalence constants depending only on  $\Omega$  and  $n$ .

*Proof.* The proof is divided into three steps.

STEP 1. We claim that  $u \in \mathcal{H}_\varepsilon(\Omega)$  if and only if  $u \in L^2(\Omega)$  and the function

$$I_y(u) = \frac{1}{y} \left[ \langle u, u \rangle_{L^2(\Omega)} - \langle u, e^{-y(-\varepsilon\Delta_N)^{1/2}} u \rangle_{L^2(\Omega)} \right]$$

is uniformly bounded in  $y > 0$ ; in such a case,

$$\lim_{y \rightarrow 0^+} I_y(u) = \sup_{y > 0} I_y(u) = \langle (-\varepsilon\Delta_N)^{1/2} u, u \rangle.$$

To prove this, observe that

$$I_y(u) = \sum_{k=0}^{\infty} \frac{1 - e^{-y(\varepsilon\lambda_k)^{1/2}}}{y} |\langle u, \varphi_k \rangle_{L^2(\Omega)}|^2 = \sum_{k=1}^{\infty} \frac{1 - e^{-y(\varepsilon\lambda_k)^{1/2}}}{y(\varepsilon\lambda_k)^{1/2}} (\varepsilon\lambda_k)^{1/2} |\langle u, \varphi_k \rangle_{L^2(\Omega)}|^2.$$

The function  $(1 - e^{-y(\varepsilon\lambda_k)^{1/2}})/(y(\varepsilon\lambda_k)^{1/2})$  converges increasingly to 1 as  $y \rightarrow 0^+$ . Then, if  $u \in \mathcal{H}_\varepsilon(\Omega)$  we have  $I_y(u) \leq \|(-\varepsilon\Delta_N)^{1/4}u\|_{L^2(\Omega)}^2$ , thus  $I_y(u)$  is uniformly bounded in  $y > 0$ . Viceversa, if  $I_y(u)$  is bounded uniformly with respect to  $y$  it clearly follows that  $u \in \mathcal{H}_\varepsilon(\Omega)$ . Apart from that, we have

$$\sup_{y > 0} I_y(u) = \lim_{y \rightarrow 0^+} I_y(u) = \sum_{k=0}^{\infty} (\varepsilon\lambda_k)^{1/2} |\langle u, \varphi_k \rangle_{L^2(\Omega)}|^2 = \langle (-\varepsilon\Delta_N)^{1/2} u, u \rangle.$$

STEP 2. For each  $u \in L^2(\Omega)$ ,

$$(2.17) \quad yI_y(u) = \frac{1}{2} \int_{\Omega} \int_{\Omega} (u(x) - u(z))^2 \mathcal{P}_{\varepsilon^{1/2}y}(x, z) dx dz, \quad y > 0,$$

where  $\mathcal{P}_y(x, z)$  is the Poisson kernel (2.12). Indeed, by the definition of  $I_y(u)$  we have

$$\begin{aligned} yI_y(u) &= \int_{\Omega} u^2(x) dx - \int_{\Omega} u(x) e^{-y(-\varepsilon\Delta_N)^{1/2}} u(x) dx \\ &= \int_{\Omega} u^2(x) dx - \int_{\Omega} \int_{\Omega} u(x) \mathcal{P}_{\varepsilon^{1/2}y}(x, z) u(z) dx dz \\ (2.18) \quad &= \int_{\Omega} \int_{\Omega} (u^2(x) - u(x)u(z)) \mathcal{P}_{\varepsilon^{1/2}y}(x, z) dx dz, \end{aligned}$$

where we have used that

$$\int_{\Omega} \mathcal{P}_y(x, z) dz = 1, \quad \text{for all } x \in \Omega, \quad y > 0,$$

which follows by applying (2.5) to (2.12). Exchanging  $x$  with  $z$  in (2.18) above, by Fubini's Theorem and the symmetry of the Poisson kernel  $\mathcal{P}_y(x, z) = \mathcal{P}_y(z, x)$ , we get

$$yI_y(u) = \int_{\Omega} \int_{\Omega} (u^2(z) - u(x)u(z)) \mathcal{P}_{\varepsilon^{1/2}y}(x, z) dx dz.$$

Thus adding this equation to (2.18) we arrive to (2.17).

STEP 3. CONCLUSION. From the subordination formula (2.12) and the two sided estimates in (2.6),

$$(2.19) \quad \mathcal{P}_{\varepsilon^{1/2}y}(x, z) \sim \frac{\varepsilon^{1/2}y}{(\varepsilon y^2 + |x - z|^2)^{\frac{n+1}{2}}}, \quad \text{for all } x, z \in \Omega, \ 0 < y < 1,$$

while  $|\mathcal{P}_{\varepsilon^{1/2}y}(x, z)| \leq M$ , for all  $x, z \in \Omega$ ,  $y > 1$ . Then, by Step 2,

$$I_y(u) \sim \int_{\Omega} \int_{\Omega} \frac{\varepsilon^{1/2}(u(x) - u(z))^2}{(\varepsilon y^2 + |x - z|^2)^{\frac{n+1}{2}}} dx dz,$$

for all  $u \in L^2(\Omega)$ . Finally, from Step 1,  $u \in \mathcal{H}_{\varepsilon}(\Omega)$  if and only if

$$\begin{aligned} \sum_{k=1}^{\infty} (\varepsilon \lambda_k)^{1/2} |\langle u, \varphi_k \rangle_{L^2(\Omega)}|^2 &= \langle (-\varepsilon \Delta_N)^{1/2} u, u \rangle = \lim_{y \rightarrow 0^+} I_y(u) \\ &\sim \varepsilon^{1/2} \int_{\Omega} \int_{\Omega} \frac{(u(x) - u(z))^2}{|x - z|^{n+1}} dx dz, \end{aligned}$$

namely, if and only if  $u \in H^{1/2}(\Omega)$ . □

**2.2. Trace inequality.** Let us define the space  $\mathcal{H}^{\varepsilon}(\mathcal{C})$  as the completion of  $H^1(\mathcal{C})$  under the scalar product

$$(2.20) \quad (v, w)_{\varepsilon} = \iint_{\mathcal{C}} (\varepsilon \nabla_x v \cdot \nabla_x w + v_y w_y) dx dy + \int_{\Omega} (\text{tr}_{\Omega} v)(\text{tr}_{\Omega} w) dx.$$

We denote by  $\|\cdot\|_{\varepsilon}$  the associated norm:

$$(2.21) \quad \|v\|_{\varepsilon}^2 = \iint_{\mathcal{C}} (\varepsilon |\nabla_x v|^2 + v_y^2) dx dy + \int_{\Omega} (\text{tr}_{\Omega} v)^2 dx.$$

Notice that, for each  $\varepsilon > 0$ ,

$$H^1(\mathcal{C}) \subset \mathcal{H}^{\varepsilon}(\mathcal{C}),$$

as Hilbert spaces, where the inclusion is strict, since constant functions belong to  $\mathcal{H}^{\varepsilon}(\mathcal{C})$  but not to  $H^1(\mathcal{C})$ . Let us point out that for a finite-height cylinder

$$\mathcal{C}_k = \Omega \times (0, k), \quad \text{for } k > 0,$$

the following trace inequality

$$(2.22) \quad \|v\|_{L^2(\mathcal{C}_k)}^2 \leq C \left( \|\text{tr}_{\Omega} v\|_{L^2(\Omega)}^2 + \|\nabla_{x,y} v\|_{L^2(\mathcal{C}_k)}^2 \right),$$

holds for all  $v \in H^1(\mathcal{C}_k)$ , where the constant  $C$  depends only on  $\Omega, k$  and  $n$ . Indeed, this result can be proved by contradiction and applying the Rellich–Kondrachov theorem. Then, by (2.22), we easily infer that  $H^1(\mathcal{C}_k) = \mathcal{H}^{\varepsilon}(\mathcal{C}_k)$  as Hilbert spaces. According to Remark 2.3, we notice also that in general the  $\varepsilon$ -Neumann extension  $v$  of a nonzero average function  $u \in \mathcal{H}_{\varepsilon}(\Omega)$  is in  $\mathcal{H}^{\varepsilon}(\mathcal{C})$  but *not* in  $H^1(\mathcal{C})$ . Our aim is now to show that there is a trace operator defined on the whole  $\mathcal{H}^{\varepsilon}(\mathcal{C})$ .

**Lemma 2.5** (Traces of functions in  $\mathcal{H}^{\varepsilon}(\mathcal{C})$ ). *For all  $v \in H^1(\mathcal{C})$  one has*

$$(2.23) \quad \begin{aligned} \|(-\varepsilon \Delta_N)^{1/4} v(x, 0)\|_{L^2(\Omega)}^2 &= \sum_{k=1}^{\infty} (\varepsilon \lambda_k)^{1/2} |\langle \text{tr}_{\Omega} v, \varphi_k \rangle_{L^2(\Omega)}|^2 \\ &\leq \iint_{\mathcal{C}} (\varepsilon |\nabla_x v|^2 + |v_y|^2) dx dy. \end{aligned}$$

In particular, equality holds in (2.23) if  $v = E^\varepsilon(\text{tr}_\Omega v)$ . Moreover, for each  $\varepsilon > 0$ , there is a unique bounded linear operator  $T^\varepsilon : H^\varepsilon(\mathcal{C}) \rightarrow \mathcal{H}_\varepsilon(\Omega)$  such that  $T^\varepsilon v(x) = v(x, 0)$  if  $v \in H^1(\mathcal{C})$  and, in particular,

$$(2.24) \quad \|T^\varepsilon v\|_{\mathcal{H}_\varepsilon(\Omega)} \leq \|v\|_\varepsilon.$$

*Proof.* By Theorem (2.4) we have that  $v(\cdot, 0) \in H^{1/2}(\Omega) = \mathcal{H}_\varepsilon(\Omega)$ . Take  $\tilde{u} = v(\cdot, 0) - v(\cdot, 0)_\Omega$ , and consider the function

$$\tilde{v}(x, y) = v(x, y) - v(\cdot, y)_\Omega$$

Then  $\tilde{v} \in H^1(\mathcal{C})$  and  $\tilde{v}(x, 0) = \tilde{u}(x)$ , with  $\tilde{u}_\Omega = 0$ . Thus from the fact that  $\varepsilon$ -Neumann extension  $E^\varepsilon(\tilde{u})$  of  $\tilde{u}$  is an energy minimizer (see Theorem 2.1) and formula (2.14), we have

$$\begin{aligned} \iint_{\mathcal{C}} (\varepsilon |\nabla_x v|^2 + |v_y|^2) dx dy &\geq \iint_{\mathcal{C}} (\varepsilon |\nabla_x \tilde{v}|^2 + |\tilde{v}_y|^2) dx dy \\ &\geq \iint_{\mathcal{C}} (\varepsilon |\nabla_x E^\varepsilon(\tilde{u})|^2 + |E^\varepsilon(\tilde{u})_y|^2) dx dy \\ &= \|(-\varepsilon \Delta_N)^{1/4} \tilde{u}\|_{L^2(\Omega)}^2 = \|(-\varepsilon \Delta_N)^{1/4} u\|_{L^2(\Omega)}^2 \end{aligned}$$

that is estimate (2.23). Now using (2.24), it is always possible to extend the trace operator on  $\Omega \times \{0\}$  from  $H^1(\mathcal{C})$  into  $H^\varepsilon(\mathcal{C})$ . Indeed, if  $v \in H^\varepsilon(\mathcal{C})$  and  $\{v_k\}_{k \in \mathbb{N}} \subset H^1(\mathcal{C})$  is a sequence converging to  $v$  in  $H^\varepsilon(\mathcal{C})$ , we define the operator

$$T^\varepsilon v := \lim_{k \rightarrow \infty} v_k(\cdot, 0),$$

where the limit is taken in  $\mathcal{H}_\varepsilon(\Omega)$ . Notice that this limit exists because, by inequality (2.24),

$$\|v_k(\cdot, 0) - v_l(\cdot, 0)\|_{\mathcal{H}_\varepsilon(\Omega)} \leq \|v_k - v_l\|_\varepsilon \rightarrow 0, \quad k, l \rightarrow \infty,$$

and obviously the definition of  $T^\varepsilon$  over  $H^\varepsilon(\mathcal{C})$  does not depend on the chosen sequence. Since

$$\|v_k(\cdot, 0)\|_{\mathcal{H}_\varepsilon(\Omega)} \leq \|v_k\|_\varepsilon, \quad \text{for all } k,$$

it is clear that  $T^\varepsilon$  is linear and bounded. Finally this operator is unique, due to the density of  $H^1(\mathcal{C})$  in  $H^\varepsilon(\mathcal{C})$ .  $\square$

**2.3. Compactness of the trace operator  $T^\varepsilon$ .** The following remark follows from Theorem 2.4 and Lemma 2.5.

**Remark 2.6.** For all  $u \in \mathcal{H}_\varepsilon(\Omega)$ , we have

$$\|u\|_{\mathcal{H}_\varepsilon(\Omega)} \geq C(\varepsilon) \|u\|_{H^{1/2}(\Omega)},$$

where

$$C(\varepsilon) = c \sqrt{\min\{1, \varepsilon^{1/2}\}}$$

being  $c$  a positive constant depending only on  $n$  and  $\Omega$ . In particular,

$$T^\varepsilon : H^\varepsilon(\mathcal{C}) \rightarrow H^{1/2}(\Omega),$$

and for every  $v \in H^\varepsilon(\mathcal{C})$ ,

$$(2.25) \quad C(\varepsilon) \|T^\varepsilon v\|_{H^{1/2}(\Omega)} \leq \|v\|_\varepsilon.$$

Of course from inequality (2.25) and the Sobolev embedding (see [1])

$$H^{1/2}(\Omega) \hookrightarrow L^{\frac{2n}{n-1}}(\Omega)$$

it follows that, for all  $v \in H^\varepsilon(\mathcal{C})$ ,

$$\|T^\varepsilon v\|_{L^{\frac{2n}{n-1}}(\Omega)} \leq C_0 \|T^\varepsilon v\|_{H^{1/2}(\Omega)} \leq C_0 C(\varepsilon)^{-1} \|v\|_\varepsilon.$$

In conclusion,

$$(2.26) \quad C_0^{-1} C(\varepsilon) \|T^\varepsilon v\|_{L^{\frac{2n}{n-1}}(\Omega)} \leq \|v\|_\varepsilon,$$

where the constant  $C_0$  does not depend on  $\varepsilon$ . Inequality (2.26) will be called the *trace embedding inequality* for the space  $H^\varepsilon(\mathcal{C})$ . An immediate consequence is the following compactness result for traces of functions in  $H^\varepsilon(\mathcal{C})$ , which follows from the compact embedding  $H^{1/2}(\Omega) \subset\subset L^q(\Omega)$  for  $1 \leq q < 2n/(n-1)$ .

**Corollary 2.7.** *Fix  $\varepsilon > 0$ . Then*

$$T^\varepsilon(H^\varepsilon(\mathcal{C})) \subset\subset L^q(\Omega), \quad \text{for all } 1 \leq q < 2n/(n-1).$$

### 3. THE LINEAR NEUMANN PROBLEM: REGULARITY

In this section we study regularity properties of solutions  $u$  to the linear fractional Neumann problem (1.11), where  $f \in H^{-1/2}(\Omega)$ . We understand (1.11) in the sense that

$$(3.1) \quad (-\varepsilon \Delta_N)^{1/2} u + u = f, \quad \text{in } H^{-1/2}(\Omega).$$

The linear problem (1.11) has a unique explicit solution  $u$  in  $H^{1/2}(\Omega) = \mathcal{H}_\varepsilon(\Omega)$  which is given in terms of the Neumann eigenfunctions of the Laplacian (2.1). Indeed, by virtue of Theorem 2.4, we see that  $H^{-1/2}(\Omega)$  coincides with the dual space of  $\mathcal{H}_\varepsilon(\Omega)$ , for each  $\varepsilon > 0$ . Then the Riesz representation theorem implies that  $f \in \mathcal{H}_\varepsilon(\Omega)'$  can be written in a unique way as  $f = \sum_{k=0}^\infty f_k \varphi_k$  in  $H^{-1/2}(\Omega)$ , where  $\sum_{k=0}^\infty ((\varepsilon \lambda_k)^{1/2} + 1)^{-1} f_k^2 < \infty$ . Then the solution  $u$  is given by

$$(3.2) \quad u(x) = \sum_{k=0}^\infty \frac{f_k}{(\varepsilon \lambda_k)^{1/2} + 1} \varphi_k(x) \in H^{1/2}(\Omega).$$

It is readily verified that  $u$  solves (1.11) and it is unique because of the orthogonality of the eigenfunctions  $\varphi_k$ .

**Remark 3.1** (Regularity in  $H^1(\Omega)$ ). If  $f \in L^2(\Omega)$ , the solution  $u$  to (1.11) is actually in  $H^1(\Omega)$ . Indeed, according to (3.2) we have

$$u = \sum_{k=0}^\infty \frac{\langle f, \varphi_k \rangle_{L^2(\Omega)}}{(\varepsilon \lambda_k)^{1/2} + 1} \varphi_k,$$

and

$$\sum_{k=1}^\infty \lambda_k \frac{|\langle f, \varphi_k \rangle_{L^2(\Omega)}|^2}{|(\varepsilon \lambda_k)^{1/2} + 1|^2} \leq \frac{1}{\varepsilon} \sum_{k=1}^\infty |\langle f, \varphi_k \rangle_{L^2(\Omega)}|^2 < \infty.$$

Hence, by the spectral representation of  $H^1(\Omega)$  provided by (2.2),  $u \in H^1(\Omega)$ .

By virtue of the extension problem for  $(-\varepsilon \Delta_N)^{1/2}$ , we can show that  $u$  is the trace over  $\Omega$  of the solution  $v$  to the following extension problem:

$$(3.3) \quad \begin{cases} \varepsilon \Delta_x v + v_{yy} = 0, & \text{in } \mathcal{C}, \\ \partial_\nu v = 0, & \text{on } \partial_L \mathcal{C}, \\ -v_y(x, 0) + v(x, 0) = f, & \text{in } H^{-1/2}(\Omega). \end{cases}$$

Indeed, by multiplying formally the equation above by a test function and integrating by parts, we can give the following suitable definition of weak solution.

**Definition 3.2.** *Let  $f \in H^{-1/2}(\Omega)$ . We say that a function  $v \in H^\varepsilon(\mathcal{C})$  is a weak solution to (3.3) if*

$$(3.4) \quad \iint_{\mathcal{C}} (\varepsilon \nabla_x v \cdot \nabla_x w + v_y w_y) dx dy + \int_{\Omega} (\text{tr}_\Omega v)(\text{tr}_\Omega w) dx = \langle f, \text{tr}_\Omega w \rangle,$$

for every  $w \in H^\varepsilon(\mathcal{C})$ .

Notice that  $T^\varepsilon w \in H^{1/2}(\Omega)$  for all  $w \in H^\varepsilon(\mathcal{C})$  so the right hand side in (3.4) is well defined.

The following Lemma provides the explicit form of the weak solution to problem (3.3).

**Lemma 3.3.** *The function*

$$v(x, y) = \sum_{k=0}^{\infty} e^{-y(\varepsilon\lambda_k)^{1/2}} \frac{f_k}{(\varepsilon\lambda_k)^{1/2} + 1} \varphi_k(x),$$

is the unique weak solution  $v \in H^\varepsilon(\mathcal{C})$  of (3.3). Moreover,  $T^\varepsilon v$  coincides with the unique solution  $u \in H^{1/2}(\Omega)$  to the linear problem (1.11).

*Proof.* Simple computations as in the proof of Theorem 2.1 show that  $v \in H^\varepsilon(\mathcal{C})$ . Let  $w \in H^1(\mathcal{C})$  be a test function. We can write, for almost every  $y \geq 0$ ,

$$w(x, y) = \sum_{k=0}^{\infty} \langle w(\cdot, y), \varphi_k \rangle_{L^2(\Omega)} \varphi_k =: \sum_{k=0}^{\infty} w_k(y) \varphi_k(x).$$

Then by using (2.4), the definition of  $v$  and (2.3), we have for almost every  $y > 0$

$$\int_{\Omega} v_{yy} w \, dx = \sum_{k=0}^{\infty} \varepsilon \lambda_k e^{-y(\varepsilon\lambda_k)^{1/2}} \frac{f_k}{(\varepsilon\lambda_k)^{1/2} + 1} w_k(y) = \varepsilon \int_{\Omega} \nabla_x v \cdot \nabla_x w \, dx.$$

Integrating this identity in  $y$  and applying integration by parts,

$$\begin{aligned} \iint_{\mathcal{C}} (\varepsilon \nabla_x v \cdot \nabla_x w + v_y w_y) \, dx \, dy &= - \lim_{y \rightarrow 0^+} \int_{\Omega} v_y(x, y) w(x, y) \, dx \\ &= \sum_{k=0}^{\infty} (\varepsilon \lambda_k)^{1/2} \frac{f_k}{(\varepsilon \lambda_k)^{1/2} + 1} w_k(0) \\ &= \sum_{k=0}^{\infty} f_k w_k(0) - \sum_{k=0}^{\infty} \frac{f_k}{(\varepsilon \lambda_k)^{1/2} + 1} w_k(0) \\ &= \langle f, \text{tr}_{\Omega} w \rangle - \int_{\Omega} (\text{tr}_{\Omega} v)(\text{tr}_{\Omega} w) \, dx. \end{aligned}$$

Hence a density argument shows that the function  $v$  is a weak solution in the sense of Definition 3.2. Uniqueness can be proved in two ways as explained in the proof of Theorem 2.1.  $\square$

**3.1. Harnack estimates.** When studying the spike layer solutions to the semilinear problem we will need the following Harnack estimate.

**Theorem 3.4** (Harnack inequality). *Let  $u \in H^{1/2}(\Omega)$  be a nonnegative solution to*

$$\begin{cases} (-\varepsilon \Delta)^{1/2} u + c(x)u = 0, & \text{in } \Omega, \\ \partial_\nu u = 0, & \text{on } \partial\Omega, \end{cases}$$

where  $c(x)$  is a bounded function. Then for any  $R > 0$  there exists a positive constant  $C$  depending only on  $n$ ,  $\Omega$  and  $R(\|c\|_{L^\infty(\Omega)}/\varepsilon)^{1/2}$  such that for any ball  $B = B(x_0, R)$ , with  $x_0 \in \overline{\Omega}$ ,

$$\sup_{B \cap \Omega} u \leq C \inf_{B \cap \Omega} u.$$

*Proof.* The proof relies on the extension problem and the reflection method. Let  $v$  be a nonnegative solution to the extension problem

$$\begin{cases} \varepsilon \Delta_x v + v_{yy} = 0, & \text{in } \mathcal{C}, \\ \partial_\nu v = 0, & \text{on } \partial_L \mathcal{C}, \\ v_y = c(x)v(x, 0), & \text{on } \Omega. \end{cases}$$

If the ball  $B$  lies in the interior of the domain  $\Omega$ , then we can repeat the proof of [4, Lemma 2.5]. If  $x_0$  lies in the boundary of  $\Omega$ , we first extend a suitable modification of  $v$  to the whole cylinder  $\Omega \times \mathbb{R}$  as in [4, Lemma 2.5]. Then we can follow the idea of [21, Proof of Lemma 4.3]. We flatten the boundary around  $x_0$  and reflect the solution to have a solution to a linear elliptic equation on a cylinder  $B_r \times \mathbb{R}$ . The interior Harnack inequality for linear equations applies and we get the result on the boundary.  $\square$

**3.2. Regularity.** Here we establish the basic regularity properties for the linear problem (1.11). Again the use of the Neumann heat semigroup plays a central role.

**Theorem 3.5** (Regularity estimates). *Let  $u \in H^{1/2}(\Omega)$  and  $f \in H^{-1/2}(\Omega)$  be such that (1.11) holds in the sense of (3.1). Then the following assertions hold.*

(1) *Let  $1 \leq p \leq \infty$  and  $f \in L^p(\Omega)$ . Then  $u \in L^p(\Omega)$  and*

$$(3.5) \quad \|u\|_{L^p(\Omega)} \leq C_{\varepsilon, n, p, \Omega} \|f\|_{L^p(\Omega)}.$$

*Moreover:*

(a<sub>1</sub>) *if  $1 < p < n$ , then  $u \in L^q(\Omega)$  for  $p \leq q \leq \frac{np}{n-p}$ , and*

$$(3.6) \quad \|u\|_{L^q(\Omega)} \leq C_{\varepsilon, n, p, q, \Omega} \|f\|_{L^p(\Omega)};$$

(b<sub>1</sub>) *if  $n < p \leq \infty$ , then  $u \in L^\infty(\Omega)$  and*

$$(3.7) \quad \|u\|_{L^\infty(\Omega)} \leq C_{\varepsilon, n, p, \Omega} \|f\|_{L^p(\Omega)}.$$

(2) *If  $f \in L^p(\Omega)$  for  $n \leq p < \infty$ , then:*

(a<sub>2</sub>) *for  $p > n$  we have  $u \in C^{0, \alpha}(\overline{\Omega})$ , where  $\alpha := 1 - \frac{n}{p} \in (0, 1)$  and*

$$(3.8) \quad \|u\|_{C^{0, \alpha}(\overline{\Omega})} \leq C_{\varepsilon, n, p, \Omega} \|f\|_{L^p(\Omega)};$$

(b<sub>2</sub>) *for  $p = n$ , we have  $u \in BMO(\Omega)$  and*

$$(3.9) \quad \|u\|_{BMO(\Omega)} \leq C_{\varepsilon, n, p, \Omega} \|f\|_{L^p(\Omega)}.$$

(c<sub>2</sub>) *Besides, if  $p \leq r < \infty$  we have  $u \in L^r(\Omega)$  and the  $L^r$ -norm of  $u$  can be added at the left-hand side of both inequalities (3.8) and (3.9), provided the constant at the right-hand side depends on  $r$  too.*

(3) *If  $f \in L^\infty(\Omega)$  then  $u \in C^{0, \alpha}(\overline{\Omega})$  for any  $0 < \alpha < 1$ , and*

$$\|u\|_{C^{0, \alpha}(\overline{\Omega})} \leq C_{\varepsilon, n, \alpha, \Omega} \|f\|_{L^\infty(\Omega)}.$$

(4) *If  $f \in C^{0, \alpha}(\overline{\Omega})$  for  $0 < \alpha < 1$ , then  $u \in C^{1, \alpha}(\overline{\Omega})$  and*

$$\|u\|_{C^{1, \alpha}(\overline{\Omega})} \leq C_{\varepsilon, n, \alpha, \Omega} \|f\|_{C^{0, \alpha}(\overline{\Omega})}.$$

**Remark 3.6.** If the constants  $C$  in the estimates above are tracked down along the proof, we would readily see that they blow up when  $\varepsilon \rightarrow 0^+$ .

*Proof of Theorem 3.5.* Recall (3.2). If  $f$  is also locally integrable then we can write

$$(3.10) \quad \begin{aligned} u(x) &= ((-\varepsilon \Delta_N)^{1/2} + I)^{-1} f(x) = \int_0^\infty e^{-t} e^{-t(-\varepsilon \Delta_N)^{1/2}} f(x) dt \\ &= \int_0^\infty e^{-t} e^{-(t\varepsilon^{1/2})(-\Delta_N)^{1/2}} f(x) dt = \int_\Omega L(x, z) f(z) dz, \end{aligned}$$

where

$$L(x, z) = \int_0^\infty e^{-t} \mathcal{P}_{(\varepsilon^{1/2}t)}(x, z) dt, \quad x, z \in \Omega,$$

and  $\mathcal{P}_t$  is the kernel of the Poisson semigroup for the Neumann Laplacian (see (2.12)). Now, from (2.19) and by using the change of variables  $s = \frac{\varepsilon^{1/2}t}{|x-z|}$  it follows that, for some finite constant  $C_n$ ,

$$0 \leq L(x, z) \leq \frac{C_{n, \varepsilon, \Omega}}{|x - z|^{n-1}}, \quad \text{for } x, z \in \Omega.$$

The estimate above implies that

$$|u(x)| \leq C_{n, \varepsilon, \Omega} N * |f \chi_\Omega|(x),$$

where  $N(x) = |x|^{1-n}$  is the kernel of the classical fractional integral of order 1.

(1) This result follows from Young's inequality for convolutions, since  $N(x) \chi_\Omega(x) \in L^1(\mathbb{R}^n)$ . Moreover, inequality (3.6) follows from the well known Hardy–Littlewood–Sobolev inequality for fractional

integration, see [17] or [30, Chapter V]. Concerning (3.7), notice that, from (2.11) and (2.19), by Hölder's inequality with  $\frac{1}{p} + \frac{1}{q} = 1$  and by applying the change of variables  $w = z/(\varepsilon^{1/2}t)$ , we get for  $t < 1$ ,

$$\begin{aligned} |e^{-(t\varepsilon^{1/2})(-\Delta_N)^{1/2}} f(x)| &\leq C_{n,\Omega} \|f\|_{L^p(\Omega)} \left( \int_{\mathbb{R}^n} \frac{(\varepsilon^{1/2}t)^q}{((\varepsilon^{1/2}t)^2 + |z|^2)^{q\frac{n+1}{2}}} dz \right)^{1/q} \\ &= C_{\varepsilon,n,\Omega} t^{\frac{n-nq}{q}} \|f\|_{L^p(\Omega)} \left( \int_{\mathbb{R}^n} \frac{1}{(1 + |w|^2)^{q\frac{n+1}{2}}} dw \right)^{1/q} \\ &= C_{\varepsilon,n,p,\Omega} t^{-n/p} \|f\|_{L^p(\Omega)}. \end{aligned}$$

On the other hand, for  $t \geq 1$ , Hölder's inequality readily gives  $|e^{-(t\varepsilon^{1/2})(-\Delta_N)^{1/2}} f(x)| \leq C \|f\|_{L^p(\Omega)}$ . We can now estimate in the second to last identity of (3.10):

$$|u(x)| \leq C_{\varepsilon,n,p,\Omega} \|f\|_{L^p(\Omega)} \int_0^\infty e^{-t} \min\{t, 1\}^{-n/p} dt = C_{\varepsilon,n,p,\Omega} \|f\|_{L^p(\Omega)},$$

which gives the conclusion (notice that  $-n/p + 1 > 0$ ).

(2) The classical result by S. Campanato in [10] establishes that  $C^\alpha(\overline{\Omega})$  for  $\alpha \in (0, 1)$  coincides, with equivalent seminorms, with the space  $BMO^\alpha(\Omega)$ . The fact that  $u \in BMO^\alpha(\Omega)$ ,  $0 \leq \alpha < 1$ , can be established as in the classical case of the fractional integral on  $\mathbb{R}^n$ , see for example [17]. For completeness we provide the proof, where the Neumann heat kernel plays a useful role again. Let  $f$  be as in the hypothesis. By (1) it follows that  $u \in L^p(\Omega)$ , hence  $u$  is locally integrable. Let  $x_0 \in \Omega$  and let  $B$  be a ball centered at  $x_0$  of radius  $r_B$ . We decompose  $f$  as  $f = f_1 + f_2$ , where  $f_1 = f\chi_{(2B)\cap\Omega}$  and  $2B$  is the ball centered at  $x_0$  with radius  $2r_B$ . By (3.10) we get  $u(x) = u_1(x) + u_2(x)$ , where  $u_i$  corresponds to the integral of  $f_i(y)$  against  $L(x, y)$ ,  $i = 1, 2$ . We denote by  $(u)_D$  the integral mean of  $u$  over a set  $D$ . We can write

$$\begin{aligned} (3.11) \quad &\frac{1}{|B \cap \Omega|} \int_{B \cap \Omega} |u(x) - (u)_{B \cap \Omega}| dx \\ &\leq \frac{2}{|B \cap \Omega|} \int_{B \cap \Omega} |u_1(x)| dx + \frac{1}{|B \cap \Omega|} \int_{B \cap \Omega} |u_2(x) - (u_2)_{B \cap \Omega}| dx =: I + II. \end{aligned}$$

Let us choose  $1 < \gamma < n$  and  $q = \frac{n\gamma}{n-\gamma}$ . By assertion (a<sub>1</sub>) and Hölder's inequality with  $r = p/\gamma$  and  $r' = \frac{p}{p-\gamma}$ ,

$$\begin{aligned} (3.12) \quad I &\leq \left( \frac{2}{|B \cap \Omega|} \int_{B \cap \Omega} |u_1(x)|^q dx \right)^{1/q} \leq \frac{C}{|B \cap \Omega|^{1/q}} \|f_1\|_{L^\gamma(\Omega)} \\ &\leq \frac{C}{|B \cap \Omega|^{\frac{n-\gamma}{n\gamma}}} \|f\|_{L^p(B \cap \Omega)} |B \cap \Omega|^{\frac{p-\gamma}{p\gamma}} = C \|f\|_{L^p(\Omega)} |B \cap \Omega|^{\frac{1}{n} - \frac{1}{p}} \\ &= C \|f\|_{L^p(\Omega)} |B \cap \Omega|^{\alpha/n}, \end{aligned}$$

with  $C$  as in (a<sub>1</sub>) and  $\alpha = 1 - n/p$ . On the other hand,

$$(3.13) \quad II \leq \frac{1}{|B \cap \Omega|^2} \int_{B \cap \Omega} \int_{B \cap \Omega} \int_{(2B)^c \cap \Omega} |f_2(z)| |L(x, z) - L(y, z)| dz dy dx.$$

Since  $x, y \in B$  and  $z \in (2B)^c$ , by the mean value theorem applied in the definition of  $L$  (use (2.7) into the second identity of (2.12)) it follows that

$$|L(x, z) - L(y, z)| \leq C \frac{\varepsilon^{-1/2} r_B}{|x_0 - z|^n}.$$

Thus Hölder's inequality yields

$$II \leq Cr_B \int_{(2B)^c \cap \Omega} \frac{|f_2(z)|}{|x_0 - z|^n} dz \leq Cr_B \|f\|_{L^p(\Omega)} \left( \int_{(2B)^c} \frac{1}{|x_0 - z|^{\frac{np}{p-1}}} dz \right)^{\frac{p-1}{p}} = C \|f\|_{L^p(\Omega)} r_B^{1-n/p}.$$

Collecting (3.11), (3.12) and (3.13),

$$\frac{1}{|B \cap \Omega|} \int_{B \cap \Omega} |u(x) - (u)_{B \cap \Omega}| dx \leq C \|f\|_{L^p(\Omega)} |B \cap \Omega|^{\alpha/n},$$

so that  $u \in BMO^\alpha(\Omega)$ . Then using (3.7), the estimates (3.8) and (3.9) follow.

Now take any  $r \in [p, \infty)$  and let us prove that  $u \in L^r(\Omega)$ . If  $p > n$  then from (3.7) we clearly have  $\|u\|_{L^r(\Omega)} \leq C_{\varepsilon, n, p, r, \Omega} \|f\|_{L^p(\Omega)}$ . Assume now that  $f \in L^n(\Omega)$ . Therefore by (3.5) we get  $u \in L^n(\Omega)$ . On the other hand, suppose that  $r > n$ . If we pick  $p$  such that  $r = \frac{np}{n-p}$  we have  $p \in (n/2, n)$ . Then  $f \in L^p(\Omega)$ , so (3.6) gives  $u \in L^{\frac{np}{n-p}}(\Omega) = L^r(\Omega)$ .

(3) For any  $\alpha \in (0, 1)$ , take  $p > n$  such that  $\alpha = 1 - n/p$  and use (3.8).

(4) If  $f \in C^{0, \alpha}(\overline{\Omega})$ , by property (3) we have that  $u \in C^{0, \alpha}(\overline{\Omega})$ . If  $v = E^\varepsilon u$ , then the rescaled function

$$\tilde{v}(x, y) = v(x, \varepsilon^{-1/2} y)$$

solves the problem

$$\begin{cases} \Delta_{x, y} \tilde{v} = 0, & \text{in } \mathcal{C}, \\ \partial_\nu \tilde{v} = 0, & \text{on } \partial_L \mathcal{C}, \\ -\tilde{v}_y(x, 0) = h(x), & \text{on } \Omega, \end{cases}$$

where

$$h = \varepsilon^{-1/2}(f - u) \in C^{0, \alpha}(\overline{\Omega}).$$

Consider the function

$$w(x, y) = \int_0^y \tilde{v}(x, t) dt, \quad y \geq 0.$$

Then we have  $w(x, 0) = 0$ , and

$$\Delta_{x, y} w = \tilde{v}_y + \int_0^y \Delta_x \tilde{v}(x, t) dt.$$

Thus by using the equation for  $\tilde{v}$  we have  $(\Delta_{x, y} w)_y = 0$  in  $\mathcal{C}$ . This says that  $\Delta_{x, y} w$  is constant as a function of  $y$ , so we can compute it by taking its value at  $\{y = 0\}$ . Observe that

$$\Delta_{x, y} w|_{y=0} = \tilde{v}_y|_{y=0} = -h(x).$$

Hence  $w$  solves

$$(3.14) \quad \begin{cases} -\Delta_{x, y} w = h(x), & \text{in } \mathcal{C}, \\ \partial_\nu w = 0, & \text{on } \partial_L \mathcal{C}, \\ w(x, 0) = 0 & \text{on } \overline{\Omega}. \end{cases}$$

Then we aim to study the boundary regularity of  $w$ . We take a point  $(x_0, 0)$  with  $x_0 \in \partial\Omega$ . Without loss of generality we can assume that  $x_0$  is the origin and that the exterior normal to  $\partial\Omega$  at  $x_0$  is the vector  $-e_n \in \mathbb{R}^n$ . Since  $\partial\Omega$  is smooth, we can describe it near  $x_0$  as the graph of an  $(n-1)$  variables differentiable map and then flatten it with a transformation  $\Psi$ . Let us call  $z$  the new variables in the flat geometry, and let  $\bar{w}(z, y) = w(x, y)$ . Notice that when we flatten the boundary we are leaving the  $y$  variable fixed. As in [21, pp. 18–19] it can be verified that  $\bar{w}(z, y)$  satisfies the following extension problem:

$$\begin{cases} -L_z \bar{w} - \bar{w}_{yy} = \bar{h}(z), & \text{in } (B_\delta \cap \{z_n > 0\}) \times (0, \infty), \\ \partial_\nu \bar{w} = 0, & \text{on } (B_\delta \cap \{z_n = 0\}) \times [0, \infty), \\ \bar{w}(z, 0) = 0, & \text{in } B_\delta \cap \{z_n \geq 0\}, \end{cases}$$



for some small  $\delta > 0$ , where  $B_\delta = B_\delta(0)$  is the ball centered at 0 of radius  $\delta$ . Here  $L_z$  is a nondivergence form elliptic operator with smooth coefficients. At this point, we could extend  $\bar{w}$ , the operator  $L_z$  and the known term  $\bar{h}$  to  $B_\delta \times [0, \infty)$  by an even reflection to get another equation in nondivergence form posed in the whole ball  $B_\delta$ , and satisfied by the even reflection  $\bar{w}_{ev}$  of  $\bar{w}$ . Then [15, Lemma 6.18] gives that  $\bar{w}_{ev} \in C^{2,\alpha}(B_\delta \times [0, \infty))$ . This implies that  $v \in C^{1,\alpha}(\bar{\mathcal{C}})$ .  $\square$

#### 4. EXISTENCE OF NONCONSTANT LEAST ENERGY POSITIVE REGULAR SOLUTIONS TO THE FRACTIONAL NEUMANN SEMILINEAR PROBLEM

In this section we find nonconstant positive solutions  $u \in H^{1/2}(\Omega)$  to the semilinear Neumann problem (1.2)–(1.3) and study regularity properties for small  $\varepsilon$ . Here we assume (1.4), that is,  $p$  is strictly smaller than the *critical Sobolev trace exponent*  $(n+1)/(n-1)$ . As in the linear case, in general we understand (1.2) as

$$(-\varepsilon \Delta_N)^{1/2} u + u = g(u), \quad \text{in } H^{-1/2}(\Omega),$$

where  $g(t) = (t_+)^p$ , for all  $t \in \mathbb{R}$ . In order to define a solution, consider the semilinear extension problem

$$(4.1) \quad \begin{cases} \varepsilon \Delta_x v + v_{yy} = 0, & \text{in } \mathcal{C}, \\ \partial_\nu v = 0, & \text{on } \partial_L \mathcal{C}, \\ -v_y(x, 0) + v(x, 0) = g(v(x, 0)), & \text{in } H^{-1/2}(\Omega). \end{cases}$$

For such a problem we have the following suitable definition of weak solution.

**Definition 4.1.** A function  $v \in H^\varepsilon(\mathcal{C})$  is a weak solution to (4.1) if for every  $w \in H^\varepsilon(\mathcal{C})$  we have

$$(4.2) \quad (v, w)_\varepsilon = \langle f_v^\varepsilon, T^\varepsilon w \rangle,$$

where  $(\cdot, \cdot)_\varepsilon$  is the inner product in  $H^\varepsilon(\mathcal{C})$ , see (2.20), and  $f_v^\varepsilon$  is the functional in  $H^{-1/2}(\Omega)$  defined by

$$\langle f_v^\varepsilon, \varphi \rangle = \int_\Omega g(T^\varepsilon v) \varphi \, dx, \quad \text{for each } \varphi \in H^{1/2}(\Omega).$$

Observe that  $f_v^\varepsilon$  is indeed in  $H^{-1/2}(\Omega)$  because, by Hölder's inequality and the boundedness of the trace operator  $T^\varepsilon$  from  $H^\varepsilon(\mathcal{C})$  into  $L^{2n/(n-1)}(\Omega)$ ,

$$\int_\Omega |g(T^\varepsilon v)|^{\frac{2n}{n+1}} \, dx \leq \int_\Omega |T^\varepsilon v|^{\frac{2n}{n+1}p} \, dx \leq C \|T^\varepsilon v\|_{L^{\frac{2n}{n-1}}(\Omega)}^{\frac{2np}{n-1}(n+1)},$$

for some constant  $C$ .

As in the linear case, according to what we explained in the Introduction, it is natural to give the following definition of weak solution to (1.2).

**Definition 4.2.** A function  $u \in H^{1/2}(\Omega)$  is a weak solution to (1.2) if  $u = T^\varepsilon v$ , where  $v$  solves (4.1) in the sense of Definition 4.1.

We look for a nonconstant weak solution  $v$  to (4.1) as a nonconstant critical point over  $H^\varepsilon(\mathcal{C})$  of the functional  $\mathcal{I}_\varepsilon$  (see (1.6)):

$$(4.3) \quad \mathcal{I}_\varepsilon(v) = \frac{1}{2} \|v\|_\varepsilon^2 - \int_\Omega G(T^\varepsilon v) \, dx,$$

where  $\|\cdot\|_\varepsilon$  is the norm in  $H^\varepsilon(\mathcal{C})$ , see (2.21), and

$$G(t) = \int_0^t g(\xi) \, d\xi = \begin{cases} \frac{1}{p+1} t^{p+1}, & \text{if } t \geq 0, \\ 0, & \text{if } t \leq 0. \end{cases}$$

The second term in the right hand side of (4.3) is well defined because of the fractional Sobolev embedding (notice that  $2 < p+1 < 2n/(n-1)$ ). Then we have the following results, which can be seen as the fractional version of [21, Theorem 2].

**Theorem 4.3.** *There is at least one positive nonconstant solution  $v_\varepsilon \in C^{2,\alpha}(\mathcal{C}) \cap C^{1,\alpha}(\overline{\mathcal{C}})$ , for  $0 < \alpha < 1$ , to problem (4.1), provided  $\varepsilon > 0$  is sufficiently small. In this case there exists a positive constant  $C$ , depending only on  $p$  and  $\Omega$ , such that*

$$\mathcal{I}_\varepsilon(v_\varepsilon) \leq C\varepsilon^{n/2}.$$

By taking  $u_\varepsilon = T^\varepsilon(v_\varepsilon)$  above, we clearly have the following.

**Corollary 4.4.** *There exists at least one positive nonconstant solution  $u_\varepsilon \in C^{1,\alpha}(\overline{\Omega})$ , for  $0 < \alpha < 1$ , to problem (1.2), provided  $\varepsilon > 0$  is sufficiently small.*

The rest of this section is devoted to the proof of Theorem 4.3. We split the proof in several steps.

• *Proof of existence of a nonconstant critical point for  $\varepsilon > 0$  sufficiently small.* We apply the Mountain Pass Lemma by Ambrosetti–Rabinowitz [3, Theorem 2.1], see also [21] and [12, Chapter 8], to find a nonconstant critical point  $v_\varepsilon$  of the functional  $\mathcal{I}_\varepsilon$ , over the Hilbert space  $\mathbf{H}^\varepsilon(\mathcal{C})$ . Observe that  $\mathcal{I}_\varepsilon(0) = 0$ . The application of the Mountain Pass Lemma will give us a nonconstant nonzero solution  $v_\varepsilon$ . To this aim we check several points.

1. *The functional  $\mathcal{I}_\varepsilon$  is in  $C^1(\mathbf{H}^\varepsilon(\mathcal{C}); \mathbb{R})$  and  $\mathcal{I}'_\varepsilon$  is Lipschitz continuous on bounded sets of  $\mathbf{H}^\varepsilon(\mathcal{C})$ .* Indeed, we have

$$\mathcal{I}_\varepsilon(v) = \frac{1}{2}\|v\|_\varepsilon^2 - \int_{\Omega} G(T^\varepsilon v) dx = \mathcal{I}_{\varepsilon,1}(v) - \mathcal{I}_{\varepsilon,2}(v),$$

where

$$\mathcal{I}_{\varepsilon,1}(v) := \frac{1}{2}\|v\|_\varepsilon^2$$

and

$$\mathcal{I}_{\varepsilon,2}(v) := \int_{\Omega} G(T^\varepsilon v) dx.$$

It is standard to see (see for example [12]) that  $\mathcal{I}_{\varepsilon,1}$  satisfies the conditions in 1 above and, moreover,  $\mathcal{I}'_{\varepsilon,1}(v) = v$ . We must now analyze  $\mathcal{I}_{\varepsilon,2}$ . To this end let us consider the operator  $\mathcal{K}$  that maps every  $f \in H^{-1/2}(\Omega)$  into the weak solution  $v \in \mathbf{H}^\varepsilon(\mathcal{C})$  of the linear problem (3.3). Then we have that

$$\mathcal{K} : H^{-1/2}(\Omega) \rightarrow \mathbf{H}^\varepsilon(\mathcal{C})$$

is an isometry. We can check that for any  $v \in \mathbf{H}^\varepsilon(\mathcal{C})$ ,  $\mathcal{I}'_{\varepsilon,2}(v) = \mathcal{K}[g(T^\varepsilon v)]$ . In fact, using parallel arguments as in [12, Chapter 8], we have that for any  $w \in \mathbf{H}^\varepsilon(\mathcal{C})$

$$\begin{aligned} \mathcal{I}_{\varepsilon,2}(w) &= \int_{\Omega} G(T^\varepsilon v) dx + \int_{\Omega} g(T^\varepsilon v)(T^\varepsilon(w - v)) dx + R \\ &= \mathcal{I}_{\varepsilon,2}(v) + (\mathcal{K}[g(T^\varepsilon v)], w - v)_\varepsilon + R, \end{aligned}$$

where we used (3.4) and  $R$  is some remainder. As in [12, p. 484] again, by applying the trace inequality (2.26) we can conclude that  $R = o(\|w - v\|_\varepsilon)$ . The Lipschitz continuity of  $\mathcal{I}'_{\varepsilon,2}$  on bounded sets follows similarly. Thus  $\mathcal{I}_{\varepsilon,2}$  satisfies condition 1 above. Therefore

$$\mathcal{I}'_\varepsilon(v) = v - \mathcal{K}[g(T^\varepsilon v)].$$

2. *The functional  $\mathcal{I}_\varepsilon$  satisfies the Palais–Smale condition.* We have to show that if we choose a sequence  $\{v_k\}_{k \in \mathbb{N}}$  in  $\mathbf{H}^\varepsilon(\mathcal{C})$  such that  $\{\mathcal{I}_\varepsilon(v_k)\}_{k \in \mathbb{N}}$  is bounded and

$$(4.4) \quad \mathcal{I}'_\varepsilon(v_k) = v_k - \mathcal{K}[g(T^\varepsilon v_k)] \rightarrow 0$$

as  $k \rightarrow \infty$  in  $\mathbf{H}^\varepsilon(\mathcal{C})$ , then  $\{v_k\}_{k \in \mathbb{N}}$  is precompact in  $\mathbf{H}^\varepsilon(\mathcal{C})$ . To that end, let  $\eta > 0$ . We have

$$|(\mathcal{I}'_\varepsilon(v_k), w)_\varepsilon| \leq \eta \|w\|_\varepsilon,$$

for  $k$  large enough. If we choose  $w = v_k$  then, by (3.4),

$$\left| \|v_k\|_\varepsilon^2 - \int_{\Omega} g(T^\varepsilon v_k) T^\varepsilon v_k dx \right| \leq \eta \|v_k\|_\varepsilon,$$

for  $k$  large. In particular, for  $\eta = 1$ ,

$$\int_{\Omega} g(T^{\varepsilon} v_k) T^{\varepsilon} v_k \, dx \leq \|v_k\|_{\varepsilon}^2 + \|v_k\|_{\varepsilon}.$$

Since  $|\mathcal{I}_{\varepsilon}(v_k)| \leq C$  for all  $k$  and  $g(t) = t^p$  for  $t > 0$  and  $p > 1$ , we deduce

$$\|v_k\|_{\varepsilon}^2 \leq C + 2 \int_{\Omega} G(T^{\varepsilon} v_k) \, dx \leq C + \frac{2}{p+1} (\|v_k\|_{\varepsilon}^2 + \|v_k\|_{\varepsilon}).$$

Now  $2/(p+1) < 1$ , therefore  $\{v_k\}_{k=1}^{\infty}$  is bounded in  $H^{\varepsilon}(\mathcal{C})$ . Then, up to subsequences, we have

$$v_k \rightharpoonup v, \quad \text{weakly in } H^{\varepsilon}(\mathcal{C}).$$

By Corollary 2.7 we have that

$$T^{\varepsilon} v_k \rightarrow T^{\varepsilon} v, \quad \text{strongly in } L^{p+1}(\Omega).$$

Notice again here that  $p+1 < 2n/(n-1)$ . Then we find  $g(T^{\varepsilon} v_k) \rightarrow g(T^{\varepsilon} v)$  in  $H^{-1/2}(\Omega)$ , thus  $\mathcal{K}[g(T^{\varepsilon} v_k)] \rightarrow \mathcal{K}[g(T^{\varepsilon} v)]$  strongly in  $H^{\varepsilon}(\mathcal{C})$ . But then by (4.4) we conclude that

$$v_k = v_k - \mathcal{K}[g(T^{\varepsilon} v_k)] + \mathcal{K}[g(T^{\varepsilon} v_k)] \rightarrow \mathcal{K}[g(T^{\varepsilon} v)]$$

in  $H^{\varepsilon}(\mathcal{C})$ . Thus  $v = \mathcal{K}[g(T^{\varepsilon} v)]$  and condition 2 holds.

3. *The functional  $\mathcal{I}_{\varepsilon}$  satisfies the following condition: there is some  $\rho > 0$  such that  $\mathcal{I}_{\varepsilon}(v) > 0$  for  $0 < \|v\|_{\varepsilon} < \rho$ , and  $\mathcal{I}_{\varepsilon}(v) \geq \beta > 0$  for some  $\beta > 0$  and  $\|v\|_{\varepsilon} = \rho$ .* This follows from [3, Lemma 3.3]. Indeed, it suffices to show that

$$\int_{\Omega} G(T^{\varepsilon} v) \, dx = o(\|v\|_{\varepsilon}^2),$$

which is readily true because the trace Sobolev inequality (recall that  $1 < p < (n+1)/(n-1)$ ) yields

$$\left| \int_{\Omega} G(T^{\varepsilon} v) \, dx \right| \leq C \|v\|_{\varepsilon}^{p+1}.$$

4. *For a sufficiently small  $\varepsilon > 0$ , there is a nonnegative function  $\Phi \in H^{\varepsilon}(\mathcal{C})$  and positive constants  $t_0, C_0$  such that*

$$\mathcal{I}_{\varepsilon}(t_0 \Phi) = 0,$$

and

$$\mathcal{I}_{\varepsilon}(t \Phi) \leq C_0 \varepsilon^{n/2}, \quad \text{for } t \in [0, t_0].$$

Observe that  $\|t_0 \Phi\|_{\varepsilon} > \rho$ , where  $\rho$  is as in 3 above. Indeed, we can choose  $\Phi$  as

$$\Phi(x, y) = e^{-y/2} \varphi(x),$$

where  $\varphi$  is as in [21, p. 10]

$$\varphi(x) = \begin{cases} \varepsilon^{-n/2} (1 - \varepsilon^{-1/2} |x|), & \text{if } |x| < \sqrt{\varepsilon}, \\ 0 & \text{if } |x| \geq \sqrt{\varepsilon}. \end{cases}$$

We can also suppose that  $0 \in \Omega$  and that  $\varepsilon$  is sufficiently small so that  $\Phi \in H^{\varepsilon}(\mathcal{C})$ . Of course we have

$$\iint_{\mathcal{C}} |\nabla_x \Phi|^2 \, dx \, dy = \int_{\Omega} |\nabla \varphi|^2 \, dx, \quad \iint_{\mathcal{C}} |\Phi_y|^2 \, dx \, dy = \frac{1}{4} \int_{\Omega} \varphi^2 \, dx,$$

and clearly  $\text{tr}_{\Omega} \Phi(x) = \varphi(x)$ , so that

$$\int_{\Omega} |T^{\varepsilon} \Phi|^2 \, dx = \int_{\Omega} \varphi^2 \, dx.$$

Then, by following the same arguments as in [21, Lemma 2.4] if we set

$$\mathbf{g}(t) = \mathcal{I}_{\varepsilon}(t \Phi), \quad \text{for } t \geq 0,$$

it is seen that there exist  $t_1, t_2$  with  $0 < t_1 < t_2$  such that  $g'(t) < 0$  if  $t > t_1$  and  $g(t) < 0$  if  $t > t_2$ . The details of this proof are left to the interested reader. Now property 3 implies  $g(t) > 0$  for small  $t$ , thus there is  $t_0$  such that  $g(t_0) = 0$ , that is

$$\mathcal{I}_\varepsilon(t_0\Phi) = 0,$$

for small  $\varepsilon > 0$ . Thus we get that  $\|t_0\Phi\|_\varepsilon > \rho$ , where  $\rho$  is as in 3 above. Moreover, as in [21, p. 12], we have

$$(4.5) \quad \max_{t \geq 0} g(t) = \max_{0 \leq t \leq t_1} g(t) \leq C_0 \varepsilon^{n/2}$$

for some constant  $C_0 > 0$ . Hence 4 is proved.

5. *Conclusion.* We are in position to apply the Mountain Pass Lemma. Set  $E = H^\varepsilon(\mathcal{C})$ ,  $e = t_0\Phi$  and

$$\Gamma = \{\gamma \in C([0, 1]; E) : \gamma(0) = 0, \gamma(1) = e\}.$$

Since conditions 1–4 are satisfied, the Mountain Pass Lemma implies that the number

$$c = \min_{\gamma \in \Gamma} \max_{t \in [0, 1]} \mathcal{I}_\varepsilon(\gamma(t))$$

is a critical value of  $\mathcal{I}_\varepsilon$  in  $H^\varepsilon(\mathcal{C})$ . Thus there exists  $v_\varepsilon$  in  $H^\varepsilon(\mathcal{C})$  such that

$$\mathcal{I}'_\varepsilon(v_\varepsilon) = 0$$

and, by (4.5),

$$\mathcal{I}_\varepsilon(v_\varepsilon) = c \leq \max_{[0, t_0]} \mathcal{I}_\varepsilon(t\Phi) \leq C_0 \varepsilon^{n/2}.$$

It remains to prove that  $v_\varepsilon$  is nonconstant. Let us argue by contradiction. Suppose that  $v_\varepsilon = c_1$ , for some real number  $c_1$ . Then

$$\mathcal{I}_\varepsilon(v_\varepsilon) = \left( \frac{1}{2} c_1^2 - G(c_1) \right) |\Omega|.$$

Since  $v_\varepsilon$  is a critical point, we have  $\mathcal{I}'_\varepsilon(v_\varepsilon) = 0$ , which by using the equation implies that  $g(c_1) = c_1$ , and therefore  $c_1 = 1$ . So,

$$\mathcal{I}_\varepsilon(v_\varepsilon) = \left( \frac{1}{2} - \frac{1}{p+1} \right) |\Omega|.$$

This is in contradiction with the inequality  $\mathcal{I}_\varepsilon(v_\varepsilon) \leq C \varepsilon^{n/2}$ , that holds for small  $\varepsilon$ .

To summarize, we conclude that for small  $\varepsilon$  the functional  $\mathcal{I}_\varepsilon(v_\varepsilon)$  has at least one nonzero nonconstant critical point.  $\square$

• *Proof of smoothness for each  $\varepsilon > 0$  small.* We prove now that the nonconstant minimizers  $v_\varepsilon$  we found in the first part of the proof are actually classical solutions. To this aim, let  $u \in H^{1/2}(\Omega)$  be a solution to

$$(-\varepsilon \Delta_N)^{1/2} u + u = g(u),$$

for

$$1 < p < \frac{n+1}{n-1},$$

and let  $v$  be its  $\varepsilon$ -Neumann extension, solving problem (4.1). If we know that  $g(u) \in L^\infty(\Omega)$ , then Theorem 3.5 parts (3)–(4) imply that  $u \in C^{1,\alpha}(\overline{\Omega})$ . Arguing as in the proof of Theorem 3.5 part (4), we can see that  $v \in C^{2,\alpha}(\mathcal{C}) \cap C^{1,\alpha}(\overline{\mathcal{C}})$ . Indeed, take  $h = \varepsilon^{-1/2}(g(u) - u) \in C^{1,\alpha}(\overline{\Omega})$  and notice that, by interior Schauder estimates, the solution  $w$  to (3.14) is in  $C^{3,\alpha}(\mathcal{C})$ . Hence each  $v_\varepsilon$  is a classical solution. Therefore we are reduced to prove that  $u$  is bounded. We use a bootstrap argument in  $u$  with the aid of Theorem 3.5. To that end, recall the embedding  $H^{1/2}(\Omega) \subset L^{\frac{2n}{n-1}}(\Omega)$ . This gives that

$$u \in L^q(\Omega), \quad \text{for } q = \frac{2n}{n-1} > 2.$$

Then, since  $n \geq 2$ , it is clear that  $q > p$ .

Let us suppose first that  $n \geq 3$ . Then, since  $p > 1$ , we have  $p < q < np$ . Observe that, by the condition on  $p$ ,

$$n(p-1) < n \left( \frac{n+1}{n-1} - 1 \right) = \frac{2n}{n-1} = q.$$

Then

$$\theta := \frac{n}{np-q} > 1.$$

Suppose now that  $u \in L^r(\Omega)$  for some  $q \leq r < np$ . Then for the nonlinear term we have  $g(u) \in L^{r/p}(\Omega)$ . Since  $r/p \geq q/p > 1$ , and  $r/p < n$ , we have that  $(-\varepsilon \Delta_N)^{1/2} u + u \in L^\gamma(\Omega)$  ( $\gamma = r/p$ ) with  $1 < \gamma < n$ . Hence, from (3.6) in Theorem 3.5, we find  $u \in L^{\theta r}(\Omega)$  (observe that  $\theta r$  is certainly bigger than  $\gamma$  and smaller than  $\frac{n\gamma}{n-\gamma}$ , the latter because  $q \leq r$ ). Now we iterate this procedure in the following way. Choose a positive integer  $k$  for which  $\theta^k q < np < \theta^{k+1} q$ . Then repeat the same reasoning as above but choosing  $r = \theta^j q$ , for  $j = 0, 1, \dots, k$ . At the end one deduces that  $u \in L^{\theta^{k+1} q}(\Omega)$ . The fact that the nonlinear term  $g(u)$  is in  $L^{\theta^{k+1} q/p}(\Omega)$  and that such exponent is strictly bigger than  $n$  imply, by (3.7) in Theorem 3.5, that  $u$  is bounded.

Next we assume that  $n = 2$ , so  $q = 4$  and  $1 < p < 3$ . We consider now three possible cases.

**Case I.**  $p < 2$ . Then  $g(u) \in L^{4/p}(\Omega)$  and  $4/p > 2 = n$ . This says, by Theorem 3.5(c<sub>2</sub>), that  $(-\varepsilon \Delta_N)^{1/2} u + u \in L^r(\Omega)$  for some  $r > n$ . By (3.7) we obtain that  $u \in L^\infty(\Omega)$ .

**Case II.**  $p = 2$ . For the right hand side we have  $|u|^{p-1} u = u^2 \in L^2(\Omega)$ . Then, by Theorem 3.5(c<sub>2</sub>),  $u$  is in  $L^r(\Omega)$  for all  $r \geq 2$ , so (3.7) gives  $u \in L^\infty(\Omega)$ .

**Case III.**  $2 < p < 3$ . Here we have  $p < q = 4 < 2p = np$ , so we can apply the iteration as in the case  $n \geq 3$  above to get higher integrability for the right hand side that still ensures the boundedness of  $u$ .  $\square$

• *Proof of positivity for each  $\varepsilon > 0$  small.* From the bootstrap argument we have proved that  $v_\varepsilon \in C^{2,\alpha}(\mathcal{C}) \cap C^{1,\alpha}(\overline{\mathcal{C}})$  for any  $0 < \alpha < 1$ , where  $v_\varepsilon$  is a nonconstant critical point of the functional  $\mathcal{I}_\varepsilon$  in (4.3). In order to prove that  $v_\varepsilon > 0$  everywhere in  $\overline{\mathcal{C}}$ , let us choose  $v_\varepsilon^-$  in the weak formulation (4.2) of problem (4.1). Then we have

$$\iint_{\mathcal{C}} (\varepsilon |\nabla_x v_\varepsilon^-|^2 + |(v_\varepsilon^-)_y|^2) dx dy + \int_{\Omega} |u_\varepsilon^-|^2 dx = - \int_{\Omega} (u_\varepsilon^+)^p u_\varepsilon^- dx = 0.$$

Thus  $v_\varepsilon \geq 0$  in  $\mathcal{C}$  and  $u_\varepsilon \geq 0$  in  $\Omega$ . Then it suffices to use [22, Proposition 7] and [22, Remark 5] to get  $u_\varepsilon > 0$  in  $\overline{\Omega}$  and  $v_\varepsilon > 0$  in  $\overline{\mathcal{C}}$ .  $\square$

## 5. BOUNDEDNESS, SPIKE SHAPE OF SOLUTIONS AND NONEXISTENCE FOR LARGE $\varepsilon$

**5.1. Uniform boundedness for small  $\varepsilon$ .** We have shown so far that each solution  $u_\varepsilon$  to problem (1.2) is bounded for small  $\varepsilon$ . The next result proves that the family of solutions  $\{u_\varepsilon\}_{\varepsilon>0}$  (for small  $\varepsilon$ ) is, in fact, equibounded.

**Theorem 5.1.** *Let  $v_\varepsilon, u_\varepsilon$  be the nonconstant smooth positive solutions obtained by Theorem 4.3 and Corollary 4.4. Then*

$$(5.1) \quad \varepsilon \iint_{\mathcal{C}} |\nabla_x v_\varepsilon|^2 dx dy + \iint_{\mathcal{C}} |(v_\varepsilon)_y|^2 dx dy + \int_{\Omega} |u_\varepsilon|^2 dx = \int_{\Omega} |u_\varepsilon|^{p+1} dx \leq (2^{-1} - \theta)^{-1} C \varepsilon^{n/2},$$

where  $C$  is the constant of Theorem 4.3 and  $\theta = 1/(p+1)$ . In particular,  $u_\varepsilon \rightarrow 0$  in measure in  $\Omega$  as  $\varepsilon \rightarrow 0^+$ . Moreover, there is a constant  $C_1 > 0$ , depending on  $\Omega$  and  $C$ , such that

$$(5.2) \quad \sup_{\Omega} u_\varepsilon \leq C_1.$$

*Proof.* The proof employs a suitable adaptation of the arguments of [21, Corollary 2.1]. First observe that by taking  $v_\varepsilon$  in the weak formulation of (4.1) we find

$$(5.3) \quad \varepsilon \iint_{\mathcal{C}} |\nabla_x v_\varepsilon|^2 dx dy + \iint_{\mathcal{C}} |(v_\varepsilon)_y|^2 dx dy + \int_{\Omega} |v_\varepsilon(x, 0)|^2 dx = \int_{\Omega} |v_\varepsilon(x, 0)|^{p+1} dx,$$

so that

$$\mathcal{I}_\varepsilon(v_\varepsilon) = \frac{1}{2} \int_\Omega |v_\varepsilon(x, 0)|^{p+1} dx - \frac{1}{p+1} \int_\Omega |v_\varepsilon(x, 0)|^{p+1} dx = \left(\frac{1}{2} - \theta\right) \int_\Omega |v_\varepsilon(x, 0)|^{p+1} dx,$$

where

$$\theta = \frac{1}{p+1} < \frac{1}{2}.$$

Then Theorem 4.3 implies

$$\int_\Omega |v_\varepsilon(x, 0)|^{p+1} dx \leq (2^{-1} - \theta)^{-1} C \varepsilon^{n/2}.$$

From this and (5.3) we get (5.1).

To prove the uniform boundedness for small  $\varepsilon$  we apply a classical Moser iteration, see [15, 21]. Let us choose in the weak formulation of problem (4.1), see Definition 4.1, the test function

$$\psi = v_\varepsilon^{2s-1},$$

for some  $s \geq 1$ . Then we have

$$\begin{aligned} (5.4) \quad & \varepsilon \frac{2s-1}{s^2} \iint_C |\nabla_x(v_\varepsilon^s)|^2 dx dy + \frac{2s-1}{s^2} \iint_C |(v_\varepsilon^s)_y|^2 dx dy + \int_\Omega |v_\varepsilon^s(x, 0)|^2 dx \\ & = \int_\Omega |v_\varepsilon(x, 0)|^{p-1+2s} dx. \end{aligned}$$

Now, for  $\varepsilon < 1$ , using the trace inequality (2.26) (where  $C(\varepsilon) = c\varepsilon^{1/4}$ ) and the fact that  $(2s-1)/s^2 < 1$ , we find from (5.4)

$$C_0^2 c^2 \varepsilon^{1/2} \frac{2s-1}{s^2} \left( \int_\Omega |v_\varepsilon(x, 0)|^{s\nu} dx \right)^{2/\nu} \leq \int_\Omega |v_\varepsilon(x, 0)|^{p-1+2s} dx,$$

where  $\nu$  is the Sobolev trace embedding exponent

$$\nu = \frac{2n}{n-1}.$$

Since  $(2s-1)s^{-2} \geq s^{-1}$  we get

$$(5.5) \quad \left( \int_\Omega |v_\varepsilon(x, 0)|^{s\nu} dx \right)^{2/\nu} \leq \varepsilon^{-1/2} s \gamma^2 \int_\Omega |v_\varepsilon(x, 0)|^{p-1+2s} dx$$

where  $\gamma = (C_0 c)^{-1}$ . Parallel to [21, p. 14], we define two sequences  $\{s_j\}_{j=0}^\infty$  and  $\{M_j\}_{j=0}^\infty$  by setting

$$p-1+2s_0 = \nu, \quad p-1+2s_{j+1} = \nu s_j,$$

and

$$M_0 = ((2^{-1} - \theta)^{-1} \gamma^2 C)^{\nu/2}, \quad M_{j+1} = (\gamma^2 s_j M_j)^{\nu/2}.$$

In particular, we have that  $s_j > 1$  and  $s_j \rightarrow \infty$  as  $j \rightarrow \infty$ . We want to show that

$$(5.6) \quad \int_\Omega |v_\varepsilon(x, 0)|^{p-1+2s_j} dx \leq M_j \varepsilon^{n/2},$$

and

$$(5.7) \quad M_j \leq e^{ms_{j-1}},$$

for some  $m > 0$ . Let us prove (5.6) for  $j = 0$ . Using (2.26) and (5.1) one finds

$$\begin{aligned} \int_\Omega |v_\varepsilon(x, 0)|^{p-1+2s_0} dx &= \int_\Omega |v_\varepsilon(x, 0)|^\nu dx \leq \varepsilon^{-\nu/4} C_0^{-\nu} \|v_\varepsilon\|_\varepsilon^\nu \\ &\leq C_0^{-\nu} (2^{-1} - \theta)^{-\nu/2} C^{\nu/2} \varepsilon^{\nu n/4 - \nu/4} \\ &= \gamma^\nu (2^{-1} - \theta)^{-\nu/2} C^{\nu/2} \varepsilon^{\nu n/4 - \nu/4} \\ &= M_0 \varepsilon^{\nu n/4 - \nu/4} = M_0 \varepsilon^{n/2}. \end{aligned}$$

Furthermore, using (5.5) it is not difficult to show that if (5.6) holds for  $j > 0$ , then it holds for  $j + 1$  too. Also (5.7) follows from the proof of [21, Corollary 2.1]. Finally, by applying (5.5)–(5.6)–(5.7),

$$\begin{aligned} \left( \int_{\Omega} |v_{\varepsilon}(x, 0)|^{s_{j-1} \nu} dx \right)^{\frac{1}{\nu s_{j-1}}} &\leq \varepsilon^{-\frac{1}{4s_{j-1}}} (\gamma^2)^{\frac{1}{2s_{j-1}}} s_{j-1}^{\frac{1}{2s_{j-1}}} \left( \int_{\Omega} |v_{\varepsilon}(x, 0)|^{p-1+2s_{j-1}} dx \right)^{\frac{1}{2s_{j-1}}} \\ &\leq \varepsilon^{\frac{n-1}{4s_{j-1}}} (\gamma^2)^{\frac{1}{2s_{j-1}}} s_{j-1}^{\frac{1}{2s_{j-1}}} e^{m/2}. \end{aligned}$$

By letting  $j \rightarrow \infty$ , inequality (5.2) follows.  $\square$

**5.2. Shape of solutions.** We show that for small  $\varepsilon$ , the solution  $u_{\varepsilon}$  of our fractional semilinear problem given by Corollary 4.4 concentrates around some points and its graph looks like spikes on  $\overline{\Omega}$ .

**Theorem 5.2** (Shape of  $u_{\varepsilon}$ ). *For  $K = (k_1 \dots, k_n) \in \mathbb{Z}^n$  and  $l > 0$ , define the cube of  $\mathbb{R}^n$*

$$Q_{K,l} = \left\{ (x_1, \dots, x_n) \in \mathbb{R}^n : |x_i - lk_i| \leq \frac{l}{2}, 1 \leq i \leq n \right\}.$$

*For small  $\varepsilon$ , let us consider the solution  $u_{\varepsilon}$  given by Corollary 4.4 and define for all  $\eta > 0$  the upper level set of  $u_{\varepsilon}$*

$$\Omega_{\eta} = \{x \in \Omega : u_{\varepsilon}(x) > \eta\}.$$

*Then there is a positive integer  $m$  depending only on  $\Omega$ , the constant  $C$  appearing in Theorem 4.3 and  $\eta$ , such that  $\Omega_{\eta}$  is covered by at most  $m$  of the  $Q_{K,\sqrt{\varepsilon}}$  cubes.*

By checking the proof of [21, Proposition 4.1], we see that in order to prove Theorem 5.2 we only need the Harnack inequality of Theorem 3.4 and Lemma 5.4. The latter result is a consequence of the following proposition and its proof follows the lines of [21, Lemma 2.3]. Finally, the proof of Proposition 5.3 uses Theorem 5.1 with slight modifications of the arguments in the proof of [21, Proposition 2.2]. The rather cumbersome details are left to the interested reader.

**Proposition 5.3.** *Fix  $\varepsilon_0 > 0$ . Then there is a constant  $c_0 > 0$  such that*

$$\varepsilon \iint_C |\nabla_x v_{\varepsilon}|^2 dx dy + \iint_C |(v_{\varepsilon})_y|^2 dx dy + \int_{\Omega} |u_{\varepsilon}|^2 dx \geq c_0 \varepsilon^{n/2},$$

*for all  $\varepsilon \in (0, \varepsilon_0)$  and any solution  $v_{\varepsilon}$  to (4.1) whose trace is  $v_{\varepsilon}(\cdot, 0) = u_{\varepsilon}$ , which solves (1.2).*

**Lemma 5.4.** *Let  $u_{\varepsilon}$  be as in Corollary 4.4. Then*

$$\begin{aligned} m(q) \varepsilon^{n/2} &\leq \int_{\Omega} u_{\varepsilon}^q dx \leq M(q) \varepsilon^{n/2}, \quad \text{if } 1 \leq q < +\infty, \\ m(q) \varepsilon^{n/2} &\leq \int_{\Omega} u_{\varepsilon}^q dx \leq M(q) \varepsilon^{nq/2}, \quad \text{if } 0 < q < 1, \end{aligned}$$

*where  $m(q), M(q)$  are positive constants independent of  $\varepsilon$ , such that  $m(q) < M(q)$ .*

**5.3. Uniform boundedness for all  $\varepsilon > 0$ .** We have shown in Theorem 5.1 that the solutions  $u_{\varepsilon}$  determined in Corollary 4.4 are uniformly bounded. The following result shows that this uniform boundedness property can be extended to *all*  $\varepsilon$ , no matter how small they are.

**Theorem 5.5** (Uniform boundedness in  $\varepsilon > 0$ ). *There exists a positive constant  $C$  independent of  $\varepsilon$  such that for any positive solution  $u$  to (1.2) we have*

$$\sup_{\Omega} u \leq C.$$

*Proof.* The proof is based on a combination of techniques that are parallel to the arguments used in [21, Theorem 3(i)], whose roots can be tracked down to one of the famous papers by B. Gidas and J. Spruck [14]. We need to flatten the boundary of  $\Omega$  and then use a blow up technique together with a Liouville result for the fractional Neumann Laplacian.

The proof is divided in two steps. First one shows that, for a fixed  $\varepsilon_0 > 0$ , the estimate holds for all solutions of (1.2) uniformly in  $0 < \varepsilon \leq \varepsilon_0$ . The second step is to give the proof when  $\varepsilon \geq \varepsilon_0$ .

**Step 1.** Let  $\varepsilon_0 > 0$  be fixed. The proof goes by contradiction. That is, suppose there exists a sequence of positive solutions  $\{u_k\}_{k \in \mathbb{N}}$  of (1.2) corresponding to parameters  $\{\varepsilon_k\}_{k \in \mathbb{N}}$ , with  $0 < \varepsilon_k \leq \varepsilon_0$ , and a sequence of points  $P_k \in \overline{\Omega}$  such that

$$M_k := \sup_{\Omega} u_k = u_k(P_k) \rightarrow \infty, \quad \text{and} \quad P_k \rightarrow P \in \overline{\Omega}, \quad \text{as } k \rightarrow \infty.$$

Let  $v_k(x, y)$  be the solution to the extension problem (4.1) corresponding to each  $u_k$ , therefore  $v_k(x, 0) = u_k(x)$ . By Hopf's maximum principle, the maximum of  $v_k$  can lie only on  $\overline{\Omega} \times \{0\}$ , thus  $\sup_{\overline{\Omega}} v_k = v_k(P_k, 0) = M_k$ . In this step we have two cases, depending on where  $P$  lies.

**Case 1.** Suppose that  $P \in \partial\Omega$ . Without loss of generality we can assume that  $P$  is the origin and that the exterior normal to  $\partial\Omega$  at  $P$  is the vector  $-e_n \in \mathbb{R}^n$ . Arguing as in the proof of Theorem 3.5(4), we can straighten the boundary near  $P$  with a local diffeomorphism  $\Psi$ . Let us call  $z$  the new coordinates, and let  $\tilde{v}_k(z, y) = v_k(x, y)$ . As in [21, p. 19] it can be verified that  $\tilde{v}_k(z, y)$  satisfies the following extension problem:

$$(5.8) \quad \begin{cases} \varepsilon_k L_z \tilde{v}_k + (\tilde{v}_k)_{yy} = 0, & \text{in } (B_{2\delta} \cap \{z_n > 0\}) \times (0, \infty), \\ \partial_\nu \tilde{v}_k = 0, & \text{on } (B_{2\delta} \cap \{z_n = 0\}) \times [0, \infty), \\ -(\tilde{v}_k)_y(x, 0) = g(\tilde{v}_k(x, 0)) - \tilde{v}_k(x, 0), & \text{on } B_{2\delta} \cap \{z_n > 0\}, \end{cases}$$

for some small  $\delta > 0$ . Here  $L_z$  is a nondivergence form elliptic operator with smooth coefficients and no independent term acting in the  $z$ -variable only and  $B_{2\delta}$  is the ball on  $\mathbb{R}^n$  centered at the origin with radius  $2\delta$ . Let  $Q_k = \Psi(P_k) = (q'_k, \alpha_k)$ ,  $\alpha_k \geq 0$ . Since  $Q_k \rightarrow 0$ , we can assume that  $|Q_k| < \delta$ . Notice that

$$\lambda_k := \left( \frac{\varepsilon_k}{M_k^{p-1}} \right)^{1/2} \rightarrow 0, \quad \text{as } k \rightarrow \infty.$$

We have now two subcases.

**Subcase 1.1.**  $\alpha_k/\lambda_k$  remains bounded as  $k \rightarrow \infty$ . Then, up to a subsequence,  $\alpha_k/\lambda_k \rightarrow \alpha \geq 0$ . Define then the rescaled function

$$w_k(z, y) := \frac{1}{M_k} \tilde{v}_k(\lambda_k z' + q'_k, \lambda_k z_n, y), \quad z = (z', z_n) \in (B_{\delta/\lambda_k} \cap \{z_n > 0\}), \quad y > 0.$$

Observe that  $0 < w_k \leq 1$ . Then, from (5.8) we can verify that  $w_k$  satisfies the extension problem

$$\begin{cases} \tilde{L}_z^k w_k + (w_k)_{yy} = 0, & \text{in } (B_{\delta/\lambda_k} \cap \{z_n > 0\}) \times (0, \infty), \\ \partial_\nu w_k = 0, & \text{on } (B_{\delta/\lambda_k} \cap \{z_n = 0\}) \times [0, \infty), \\ -(w_k)_y(x, 0) = g(w_k(x, 0)) - M_k^{-(p-1)} w_k(x, 0), & \text{on } B_{\delta/\lambda_k} \cap \{z_n > 0\}. \end{cases}$$

The coefficients of  $\tilde{L}_z^k$  are now a rescaled and translated version of the coefficients of  $L_z$  and are uniformly bounded in  $k$ . Then the compactness arguments in [21] can be paralleled in such a way that we can extract from  $\{w_k\}_{k \in \mathbb{N}}$  a subsequence converging uniformly to a nonnegative solution  $w(z, y)$  to the extension problem

$$\begin{cases} \Delta_z w + w_{yy} = 0, & \text{in } \mathbb{R}_+^n \times (0, \infty), \\ \partial_\nu w = 0, & \text{on } \partial\mathbb{R}_+^n \times [0, \infty), \\ -w_y(x, 0) = g(w(x, 0)), & \text{on } \mathbb{R}_+^n. \end{cases}$$

Let us now extend  $w$  to  $\mathbb{R}^n \times [0, \infty)$  as  $w^*(z', z_n, y) = w(z', |z_n|, y)$ , so that  $w^*$  is a solution to

$$\begin{cases} \Delta_z w^* + w_{yy}^* = 0, & \text{in } \mathbb{R}^n \times (0, \infty), \\ -w_y^*(x, 0) = (w^*(x, 0))^p, & \text{in } \mathbb{R}^n. \end{cases}$$

The Liouville theorem of [18] (see also [20, Remark 1.4]) implies that  $w^*$  is identically zero. But this is a contradiction because

$$w(0, \dots, 0, \alpha, 0) = \lim_{k \rightarrow \infty} w_k(0, \dots, 0, \alpha_k/\lambda_k, 0) = \lim_{k \rightarrow \infty} \frac{1}{M_k} \tilde{v}_k(Q_k, 0) = 1.$$



**Subcase 1.2.**  $\alpha_k/\lambda_k$  is unbounded. We can suppose that  $\alpha_k/\lambda_k \rightarrow \infty$ . Now we define

$$w_k(z, y) = \frac{1}{M_k} \tilde{v}_k(\lambda_k z + Q_k, y).$$

Then the argument goes as in [21] with the proper modifications in the extension problem as we did in subcase 1.1, and using the Liouville theorem of [18, 20].

**Case 2.** Suppose that  $P$  is in the interior of  $\Omega$ . The scaling we need now is

$$w_k(z, y) = \frac{1}{M_k} \tilde{v}_k(\lambda_k z + P_k, y),$$

with  $\lambda_k$  as above and the argument goes as in as in Subcase 1.2. Details are left to the interested reader.

**Step 2.** If  $\varepsilon \geq \varepsilon_0$ , by arguing by contradiction to fall into Step 1, we can prove that

$$(5.9) \quad \sup_{\Omega} u \leq C \varepsilon^{1/2(p-1)}$$

being  $C$  a constant independent on  $\varepsilon$ . From (5.4) and (5.9) we obtain

$$(5.10) \quad \begin{aligned} & \varepsilon \frac{2s-1}{s^2} \iint_{\mathcal{C}} |\nabla_x(v^s)|^2 dx dy + \frac{2s-1}{s^2} \iint_{\mathcal{C}} |(v^s)_y|^2 dx dy + \int_{\Omega} |u^s|^2 dx = \int_{\Omega} |u|^{p-1+2s} dx \\ & \leq C^{p-1} \varepsilon^{1/2} \int_{\Omega} u^{2s} dx. \end{aligned}$$

Using Lemma 2.5 (inequality (2.23)) and Theorem 2.4 we have

$$\varepsilon \iint_{\mathcal{C}} |\nabla_x(v^s)|^2 dx dy + \iint_{\mathcal{C}} |(v^s)_y|^2 dx dy \geq \|(-\varepsilon \Delta_N)^{1/4} u^s\|_{L^2(\Omega)}^2 \geq C_1 \varepsilon^{1/2} [u^s]_{H^{1/2}(\Omega)}^2.$$

Plugging this estimate into (5.10) and noticing that  $s^2/(2s-1) \leq s$  for  $s \geq 1$ , we get

$$[u^s]_{H^{1/2}(\Omega)}^2 \leq C_2 s \int_{\Omega} u^{2s} dx.$$

Thus

$$(5.11) \quad \|u^s\|_{H^{1/2}(\Omega)}^2 = \|u^s\|_{L^2(\Omega)}^2 + [u^s]_{H^{1/2}(\Omega)}^2 \leq C_3 s \int_{\Omega} u^{2s} dx$$

where the constants  $C_i$ ,  $i = 1, 2, 3$  depend only on  $n$  and  $\Omega$ . Hence, by applying the fractional Sobolev embedding  $H^{1/2}(\Omega) \hookrightarrow L^\nu(\Omega)$  in (5.11), where

$$\nu = \frac{2n}{n-1},$$

we find

$$(5.12) \quad \left( \int_{\Omega} u^{s\nu} dx \right)^{2/\nu} \leq C_4 s \int_{\Omega} u^{2s} dx,$$

for some constant  $C_4 = C_4(n)$ . At this point we are ready to proceed as in [21, pp. 21–22]. Set

$$r_1 = p, \quad r_{j+1} = 2^{-1} \nu r_j,$$

so that

$$r_j = p (2^{-1} \nu)^{j-1},$$

and put

$$\alpha_j = \int_{\Omega} u^{r_j} dx, \quad j \geq 1.$$

Then, by (5.12),

$$\alpha_{j+1} \leq (C_5 r_j)^{\nu/2} \alpha_j^{\nu/2},$$

where  $C_5 = C_4/2$  and, as in [21, p. 21],

$$\limsup_{j \rightarrow \infty} r_j^{-1} \log \alpha_j \leq \frac{1}{p} [\log \alpha_1 + \nu^* (\nu^* - 1)^{-1} \{\log(p C_4) + (\nu^* - 1)^{-1} \log \nu^*\}]$$

where  $\nu^* = \nu/2$ . Thus

$$\|u\|_{L^\infty(\Omega)} \leq C_6 \alpha_1^{1/p},$$

for some suitable constant  $C_6 = C_6(n, \Omega)$ . By integrating the first equation in (4.1) over  $\mathcal{C}$ , we have

$$\int_{\Omega} u \, dx = \int_{\Omega} u^p \, dx.$$

Hence, by Hölder's inequality,

$$\int_{\Omega} u^p \, dx \leq |\Omega|^{(p-1)/p} \left( \int_{\Omega} u^p \, dx \right)^{1/p},$$

so

$$\alpha_1 = \int_{\Omega} u^p \, dx \leq |\Omega|,$$

namely

$$\|u\|_{L^\infty(\Omega)} \leq C_6 |\Omega|^{1/p},$$

and the proof is complete.  $\square$

**5.4. Nonexistence for large  $\varepsilon$ .** As a consequence of the boundedness result contained in Theorem 5.5 and by following ideas contained in [23] we are able now to show that  $u \equiv 1$  is actually the only positive solution to (1.2) for large  $\varepsilon$ .

**Theorem 5.6.** *There exists  $\varepsilon^* > 0$  such that if  $\varepsilon > \varepsilon^*$ , then  $u \equiv 1$  is the only positive solution to (1.2).*

*Proof.* Let  $u$  be a positive solution to (1.2) and write  $u = \phi + u_{\Omega}$ , where  $u_{\Omega}$  is as in (1.10), so that

$$(5.13) \quad \int_{\Omega} \phi \, dx = 0.$$

Then  $\phi$  satisfies the equation (recall that  $g(u) = u^p$  when  $u > 0$ )

$$(-\varepsilon \Delta_N)^{1/2} \phi + \phi - \left( \int_0^1 p(u_{\Omega} + t\phi)^{p-1} dt \right) \phi = u_{\Omega}^p - u_{\Omega}.$$

Let  $v^{\phi} = E^{\varepsilon}(\phi)$  be the  $\varepsilon$ -Neumann extension of  $\phi$ , which satisfies the extension problem

$$(5.14) \quad \begin{cases} \varepsilon \Delta_x v^{\phi} + v_{yy}^{\phi} = 0, & \text{in } \mathcal{C}, \\ \partial_{\nu} v^{\phi} = 0, & \text{on } \partial_L \mathcal{C}, \\ -\lim_{y \rightarrow 0} v_y^{\phi}(\cdot, y) = \left( \int_0^1 p(u_{\Omega} + t\phi)^{p-1} dt \right) \phi - \phi + u_{\Omega}^p - u_{\Omega}, & \text{on } \Omega. \end{cases}$$

Taking  $v^{\phi}$  as a test function in (5.14) and using (5.13) we find the identity

$$\varepsilon \iint_{\mathcal{C}} |\nabla_x v^{\phi}|^2 \, dx \, dy + \iint_{\mathcal{C}} |v_y^{\phi}|^2 \, dx \, dy + \int_{\Omega} \phi^2 \, dx = \int_{\Omega} \left( \int_0^1 p(u_{\Omega} + t\phi)^{p-1} dt \right) \phi^2 \, dx.$$

Since by Theorem 5.5 we have

$$\sup_{\Omega} u \leq C$$

where  $C$  is a constant not depending on  $\varepsilon$ , we find

$$\varepsilon \iint_{\mathcal{C}} |\nabla_x v^{\phi}|^2 \, dx \, dy + \iint_{\mathcal{C}} |v_y^{\phi}|^2 \, dx \, dy + \int_{\Omega} \phi^2 \, dx \leq p C^{p-1} \int_{\Omega} \phi^2 \, dx.$$

Thus inequality (2.23) yields

$$\|(-\varepsilon \Delta_N)^{1/4} \phi\|_{L^2(\Omega)}^2 + \int_{\Omega} \phi^2 \, dx \leq p C^{p-1} \int_{\Omega} \phi^2 \, dx,$$

which in turn implies, by Theorem 2.4, that for some constant  $C_1 > 0$ ,

$$(5.15) \quad \varepsilon^{1/2} C_1 [\phi]_{H^{1/2}(\Omega)}^2 + \int_{\Omega} \phi^2 dx \leq p C^{p-1} \int_{\Omega} \phi^2 dx.$$

Now we recall the *fractional Poincaré inequality* (see [1]) which says that there is a constant  $C_2 > 0$  such that for all  $\psi \in H^{1/2}(\Omega)$  one has

$$C_2 \|\psi - \psi_{\Omega}\|_{L^2(\Omega)} \leq [\psi]_{H^{1/2}(\Omega)}.$$

Then applying such inequality to  $\psi = \phi$ , by recalling (5.13) and inserting it into (5.15) we finally find

$$(C_1 C_2 \varepsilon^{1/2} + 1) \int_{\Omega} \phi^2 dx \leq p C^{p-1} \int_{\Omega} \phi^2 dx,$$

which is impossible if  $\varepsilon > [(p C^{p-1} - 1)/C_1 C_2]_+^2 =: \varepsilon^*$  and  $\phi \not\equiv 0$ . Then for  $\varepsilon > \varepsilon^*$  we must have  $\phi \equiv 0$ , namely  $u = u_{\Omega}$  and (1.2) implies  $u \equiv 1$ .  $\square$

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